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Brief Paper

Distributed Nash equilibrium seeking in networked graphical games ☆

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ABSTRACT

This paper considers a gossip approach for finding a Nash equilibrium in networked games on graphs, where a player's cost function may be affected by the actions of any subset of players. An interference graph illustrates the partially-coupled cost functions, i.e., the asymmetric strategic interaction and information requirements. An algorithm is proposed whereby players make decisions based only on the estimates of their interfering players' actions. Given the interference graph (not necessarily complete), a communication graph is designed so that players exchange only their required information. When the interference graph is sparse, the algorithm can offer substantial savings in communication and computation. Almost sure convergence to a Nash equilibrium is proved for diminishing step sizes. The effect of the second largest eigenvalue of the expected communication matrix on the convergence rate is quantified.

1. Introduction

Distributed seeking of Nash equilibria in networked games has received considerable attention in recent years, (Stankovic, Johansson, & Stipanovic, 2012). A networked game can be represented by a *graphical model* where the cost function of each player is indexed as a function of player's own action and those of his neighbours. A graphical game is succinctly represented via an undirected graph called *interference graph*, where players are marked by vertices and interferences by edges, (Nisan, Roughgarden, Tardos, & Vazirani, 2007; Kearns, Littman, & Singh 2001). There are many real-world applications that motivate us to generalize the Nash seeking problem to a graphical game setup, (Chen & Huang, 2012). For instance, the collection of transmitters and receivers in a wireless data network, (Alpcan & Başar, 2004), or the set of channels in an optical network, (Pavel, 2012), can be described by a graphical model.

Literature review. In this work we are interested in developing a locally distributed algorithm for Nash equilibrium seeking in graphical games, where cost functions are partially-coupled. The idea of a graphical game has been used in various areas. In congestion games, Tekin, Liu, Southwell, Huang, and Ahmad (2012) consider a *conflict graph* to specify players that cause congestion to each other. In Abouheaf and Lewis (2014), graphical games are considered in the context of dynamical games. A stronger definition of an interactive Nash equilibrium is used to guarantee a unique Nash equilibrium. Moreover, the information flow is described by a communication graph which is *identical* to the interference graph. In an economic setting, Bramoullé, Kranton, and D'Amours (2014) draw attention to the problem of "who interacts with whom" in a network and to the importance of communication with neighbouring players. The effect of local peers on increasing the usage level of consumers is addressed in Candogan, Bimpikis, and Ozdaglar (2012). Using word-of-mouth communication, players form their opinions about the quality of a product and improve their purchasing behaviour. The problem of finding a Nash equilibrium in generalized convex games where the interference graph is *not necessarily complete* is studied in Zhu and Frazzoli (2016). However, the communication graph is *identical* to the interference graph. A connected communication graph is considered in Koshal, Nedic, and Shanbhag (2012) for the special class of *aggregative* games. The coupling to others is via a common aggregated variable, hence with a *complete* interference graph. A larger class of convex games is considered in Salehisadaghiani and Pavel (2016), still with a *complete* interference graph. An asynchronous gossip-based projected-gradient algorithm is proposed over a connected communication graph.

Contributions. In this paper, we consider a class of networked games on graphs where the interference graph is *not necessarily complete*, and the connected communication graph is a *subset* of the interference graph. We adapt the algorithm in Salehisadaghiani and Pavel (2016) to account for partial-coupling. As opposed to

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Salehisadaghiani and Pavel (2016), where each player keeps full local estimates of all others' actions, herein each player maintains an estimate of *only* the actions of players interfering with him. This can greatly reduce the communication and computation overhead when the interference graph is sparse. However, due to non-uniformity in the size of estimates, the communication graph needs to be designed such that each player obtains all his required information from his communication neighbours. In this sense, we show that there exists a lower bound for the communication graph. Inspired by Boyd, Ghosh, Prabhakar, and Shah (2006) and Duchi, Agarwal, and Wainwright (2012), we show that the convergence time depends on the second largest eigenvalue of the expected communication matrix, hence depends on the communication and interference graphs. Our techniques are also similar to those used in the literature on distributed optimization (Johansson, 2008; Nedic, 2011; Nedic & Ozdaglar, 2009). However, there are technical challenges due to the game context. In distributed optimization, all agents minimize an aggregate cost function with respect to a common optimization variable. In a game setup, each player controls only his own action, which is an element of the full decision vector. Moreover, his cost function depends on the actions of a subset of other players. This translates into asymmetry and non-uniformity in players' data size and overall data exchange. We circumvent these issues by introducing generalized weight matrices and exploiting their properties to prove convergence to Nash equilibrium.

The paper is organized as follows. Problem statement and assumptions are given in Section 2. The algorithm is described in Section 3 and its convergence is shown in Section 4. The convergence rate is analysed in Section 5 and simulation results are given in Section 6.

1.1. Notations

All vector norms $\|\cdot\|$ are Euclidean. The cardinality of a set A is denoted by $|A|$. The Euclidean projection of x onto the set K is denoted by $T_K[x]$. We denote by $[a_i]_{i=1,\dots,N}$ the $N\times 1$ vector with a_i as the ith entry. We denote by $\mathbf{1}_N$ the $N\times 1$ all-ones vector and by $\mathbf{0}_N$ the all-zeros $N\times 1$ vector. We use e_i to denote a unit vector whose ith element is 1 and the others are 0. The $N\times N$ identity matrix is denoted by I_N.

The following are from Godsil and Royle (2013) and Goddard and Kleitman (1993). An undirected graph G is a pair (V,E) with V a set of vertices and $E\subseteq V\times V$ a set of edges such that for $i,j\in V$, if $(i,j)\in E$, then $(j,i)\in E$. The degree of vertex i, denoted by $\deg_G(i)$, is the number of edges connected to i. A path is a sequence of edges which connects a sequence of vertices. A graph is connected if there is a path between every pair of vertices. An adjacency matrix $A=[a_{ij}]_{i,j\in V}$ is a matrix with $a_{ij}=1$ if $(i,j)\in E$ and $a_{ij}=0$ otherwise. A subgraph H of G is a graph whose vertices and edges are a subset of the vertex and edge set of G. A supergraph H of G is a graph of which G is a subgraph. H is a spanning subgraph of G, if it contains all the vertices of G. A triangle-free spanning subgraph H of G is a subgraph in which no three vertices form a triangle of edges. H is a maximal triangle-free spanning subgraph of G if adding an edge from $G-H$ to H creates only one triangle.

2. Problem statement

Consider a game between a set of players $V=\{1,\dots,N\}$ in a network. Each player $i\in V$ has a real-valued cost function J_i, which may be affected by the actions of any number of players. To illustrate the strategic interaction between agents and the partially coupled cost functions, we define an *interference graph*, denoted by $G_I(V,E_I)$. In general G_I is not complete; the edge set E_I marks player pairs that interfere one with another. We denote by $N_I(i):=\{j\in V|(i,j)\in E_I\}$ the set of neighbours in G_I of player i, and $\tilde{N}_I(i):=N_I(i)\cup\{i\}$.

Assumption 1. The interference graph G_I is connected and undirected.

Let $\Omega_i\subset\mathbb{R}$ denote player i's action set, and $\Omega=\prod_{i\in V}\Omega_i\subset\mathbb{R}^N$ all players' action set, where $\prod$ is the Cartesian product. Player i's cost function is $J_i:\Omega^i\to\mathbb{R}$, where $\Omega^i=\prod_{j\in\tilde{N}_I(i)}\Omega_j\subset\mathbb{R}^{|\tilde{N}_I(i)|}$ is the action set of players interfering with him. Let $x_i\in\Omega_i$ be player i's action, $x^i_{-i}\in\Omega^i_{-i}:=\prod_{j\in N_I(i)}\Omega_j$ the other players' actions which affect his cost, and $x^i=(x_i,x^i_{-i})\in\Omega^i$. Let $x=(x_i,x_{-i})\in\Omega$ be all players' action profile, where $x_{-i}\in\Omega_{-i}:=\prod_{j\in V/\{i\}}\Omega_j$ denotes all other players' actions except i. The game thus defined on G_I is denoted by $\mathcal{G}(V,\Omega_i,J_i,G_I)$. For any given $x^i_{-i}\in\Omega^i_{-i}$, each player i aims to minimize his own cost function selfishly to find an optimal action,

$$\begin{aligned}\underset{y_i}{\text{minimize}}\quad & J_i(y_i,x^i_{-i})\\ \text{subject to}\quad & y_i\in\Omega_i.\end{aligned}\tag{1}$$

Note that there are N separate simultaneous optimization problems, each run by a particular player i.

Definition 1. Consider an N-player game $\mathcal{G}(V,\Omega_i,J_i,G_I)$, each player i minimizing the cost function $J_i:\Omega^i\to\mathbb{R}$. A vector $x^*=(x_i^*,x_{-i}^*)\in\Omega$ is called a Nash equilibrium of this game if for every given $x^{i*}_{-i}:=[x_j^*]_{j\in N_I(i)}\in\Omega^i_{-i}$,

$$J_i(x_i^*,x^{i*}_{-i})\le J_i(x_i,x^{i*}_{-i})\quad\forall x_i\in\Omega_i,\ \forall i\in V.\tag{2}$$

Definition 1 is a restatement of a Nash equilibrium definition so that when G_I is not complete, $J_i(x_i,x_{-i}),J_i(x_i^*,x_{-i}^*)$ are replaced with $J_i(x_i,x^i_{-i}),J_i(x_i^*,x^{i*}_{-i})$. We make the following assumptions.

Assumption 2. For every $i\in V$, the action set Ω_i is a non-empty, compact and convex subset of $\mathbb{R}$. $J_i(x_i,x^i_{-i})$ is a continuously differentiable function in x_i, jointly continuous in x^i and convex in x_i for every x^i_{-i}.

The compactness of Ω implies that $\forall i\in V$ and $x^i\in\Omega^i$,

$$\|\nabla_{x_i}J_i(x^i)\|\le C,\quad\text{for some } C>0.\tag{3}$$

Assumption 3. The pseudo-gradient $F:\Omega\to\mathbb{R}^N$, $F(x):=[\nabla_{x_i}J_i(x^i)]_{i\in V}$, is strictly monotone, i.e.,

$$(F(x)-F(y))^T(x-y)>0\quad\forall x,y\in\Omega,\ x\neq y.\tag{4}$$

Assumptions 2 and 3 imply that Nash equilibrium exists and is unique.

Assumption 4. For every $i\in V$, $\nabla_{x_i}J_i(x_i,u)$ is Lipschitz continuous in x_i, for every fixed $u\in\Omega^i_{-i}$, i.e.,

$$\|\nabla_{x_i}J_i(x_i,u)-\nabla_{x_i}J_i(y_i,u)\|\le\sigma_i\|x_i-y_i\|\ \forall x_i,y_i\in\Omega_i.\tag{5}$$

for some $\sigma_i>0$. Moreover, $\nabla_{x_i}J_i(x_i,u)$ is Lipschitz in u with Lipschitz constant $L_i>0$ for every fixed $x_i\in\Omega_i$.

Assumption 4 implies that $F(x)$ is Lipschitz continuous in $x\in\Omega$ with $\rho=\sqrt{\sum_{i\in V}\rho_i^2}$, $\rho_i^2=2L_i^2+2\sigma_i^2$.

We assume that J_i and the action set Ω^i are available to player i. Thus every player knows which other players' affect his cost function, but not his actions. We assume that each player maintains an *action estimate of only* his interfering players according to G_I, and that players exchange information over a *communication graph* $G_C(V,E_C)$. As no unnecessary data needs to be exchanged, this can