Brief Paper

# Strong stability of a class of difference equations of continuous time and structured singular value problem* 

Qian Ma ${ }^{\text {a }}$, Keqin Gu ${ }^{\text {b }}$, Narges Choubedar ${ }^{\text {b }}$<br>${ }^{\text {a }}$ School of Automation, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China<br>${ }^{\mathrm{b}}$ Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, Edwardsville, IL 62025, USA

## A R T I C L E I N F O

## Article history:

Received 25 January 2017
Received in revised form 23 July 2017
Accepted 7 September 2017

## Keywords:

Stability
Time delay
Difference equation
Structured singular value


#### Abstract

This article studies the strong stability of scalar difference equations of continuous time in which the delays are sums of a number of independent parameters $\tau_{i}, i=1,2, \ldots, K$. The characteristic quasipolynomial of such an equation is a multilinear function of $e^{-\tau_{i} s}$. It is known that the characteristic quasipolynomial of any difference equation set in the form of one-delay-per-scalar-channel (ODPSC) model is also in such a multilinear form. However, it is shown in this article that some multilinear forms of quasipolynomials are not characteristic quasipolynomials of any ODPSC difference equation set. The equivalence between local strong stability, the exponential stability of a fixed set of rationally independent delays, and the stability for all positive delays is shown, and relations with the structured singular value problem are presented. A procedure to determine strong stability in the special case of up to three independent delay parameters in finite steps is developed. This procedure means that the structured singular value problem in the case of up to three scalar complex uncertain blocks can be solved in finite steps.


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## 1. Introduction

This article studies the stability problem of systems with characteristic quasipolynomial,

$$
\begin{align*}
& \Delta(s)=1+ \\
& \sum_{m=1}^{K} \sum_{1 \leq i_{1}<i_{2}<\cdots<i_{m} \leq K} a_{i_{1} i_{2} \ldots i_{m}} e^{-\left(\tau_{i_{1}}+\tau_{i_{2}}+\cdots+\tau_{i_{m}}\right) s}, \tag{1}
\end{align*}
$$

where $\tau_{i}, i=1,2, \ldots, K$ are independent parameters, and $a_{i_{1} i_{2} \ldots i_{m}}$ are real coefficients. For $K=1,2$ and $3, \Delta(s)$ in (1) becomes
$\Delta(s)=1+a_{1} e^{-\tau_{1} s}$,
$\Delta(s)=1+a_{1} e^{-\tau_{1} s}+a_{2} e^{-\tau_{2} s}+a_{12} e^{-\left(\tau_{1}+\tau_{2}\right) s}$,

[^0]\[

$$
\begin{align*}
\Delta(s)= & 1+a_{1} e^{-\tau_{1} s}+a_{2} e^{-\tau_{2} s}+a_{3} e^{-\tau_{3} s} \\
& +a_{12} e^{-\left(\tau_{1}+\tau_{2}\right) s}+a_{13} e^{-\left(\tau_{1}+\tau_{3}\right) s} \\
& +a_{23} e^{-\left(\tau_{2}+\tau_{3}\right) s}+a_{123} e^{-\left(\tau_{1}+\tau_{2}+\tau_{3}\right) s} \tag{4}
\end{align*}
$$
\]

respectively. Obviously, $\Delta(s)$ in (1) is the characteristic quasipolynomial of the difference equation of continuous time,

$$
\begin{align*}
& y(t)+ \\
& \sum_{m=1}^{K} \sum_{1 \leq i_{1}<i_{2}<\cdots<i_{m} \leq K} a_{i_{1} i_{2} \ldots i_{m}} y\left(t-\tau_{i_{1}}-\tau_{i_{2}}-\cdots-\tau_{i_{m}}\right) \\
& =0 . \tag{5}
\end{align*}
$$

As $\Delta(s)$ in (1) is a multilinear function of $e^{-\tau_{i} s}, i=1,2, \ldots, K$, it is closely related to the following form of one-delay-per-scalarchannel (ODPSC) difference equation set,
$y_{k}(t)=\sum_{j=1}^{K} d_{k j} y_{j}\left(t-\tau_{j}\right), k=1,2, \ldots, K$,
where
$y_{k}(t) \in \mathbb{R}, d_{k j} \in \mathbb{R}, k, j=1,2, \ldots, K$.
Indeed, the characteristic function of (6) is

$$
\begin{equation*}
\Delta_{1}(s)=\operatorname{det}(I-D E)=0, \tag{7}
\end{equation*}
$$

where
$D=\left(d_{i j}\right)_{K \times K}$,
$E=\operatorname{diag}\left(e^{-\tau_{1} s}, e^{-\tau_{2} s}, \ldots, e^{-\tau_{K} s}\right)$.
An expansion of the determinant shows that $\Delta_{1}(s)$ is indeed a multilinear function of $e^{-\tau_{i} s}, i=1,2, \ldots, K$ in the form of (1). Hale and Verduyn Lunel (1993) in Section 9.6 illustrated through an example how to rewrite the difference equation of the form (5) to the ODPSC difference equation set of the form (6) for the case of $K=2$. Unfortunately, while such rewriting is always possible for $K \leq 2$, it may not be possible in some cases with $K \geq 3$ as will be shown later in this article. Therefore, studying (1) indeed has independent interest.

Difference equation of continuous time, in addition to its independent interest, also plays an important role in the theory of timedelay systems of neutral type (Gu, 2012; Hale \& Verduyn Lunel, 1993). Especially, a necessary condition for the exponential stability of the coupled differential-difference equation (8)-(9) below is the exponential stability of the associated difference equation (6).
$\dot{x}(t)=A x(t)+\sum_{j=1}^{K} B_{j} y_{j}\left(t-\tau_{j}\right)$,
$y_{k}(t)=C_{k} x(t)+\sum_{j=1}^{K} d_{k j} y_{j}\left(t-\tau_{j}\right), k=1,2, \ldots, K$,
where
$x(t) \in \mathbb{R}^{n}, y_{k}(t) \in \mathbb{R}$.
Similarly, the exponential stability of the difference equation (5) is a necessary condition for the exponential stability of the differential-difference equations of neutral type studied in Naghnaeian and Gu (2013) for $K=2$ and Gu \& Zheng (2014) for $K=3$. Time-delay systems of neutral type may arise in natural systems (Hale \& Verduyn Lunel, 1993), or as a result of feedback control such as Smith predictor (Palmor, 1980) and discrete implementation of distributed-delay feedback control (Michiels, Mondié, Roose, \& Dambrine, 2004; Mirkin, 2004; Mondié, Dambrine, \& Santos, 2002; Zhong, 2004).

The stability of difference equations of continuous time has been studied using the Lyapunov functional approach (Pepe, 2003; Shaikhet, 2011) and frequency domain approach (Avellar \& Hale, 1980; Hale \& Verduyn Lunel, 2002; Henry, 1974). This article uses the frequency domain approach. Similar to systems described by differential equations, a system described by difference equation (5) is exponentially stable if and only if all its characteristic roots $s_{k}, k=1,2, \ldots$, i.e., the solutions of the equation
$\Delta(s)=0$,
satisfy $\operatorname{Re}\left(s_{k}\right) \leq-\epsilon$ for some $\epsilon>0$.
In this article, we concentrate on the strong stability of the system (1). In other words, we are interested in the stability of (1) when the delay parameters $\tau_{1}, \tau_{2}, \ldots, \tau_{K}$ are subject to independent, although arbitrarily small, deviation from the nominal values. The surprisingly significant impact of such small deviation was first documented by Henry (1974) and Melvin (1974). Our results are analogous to the one given by Hale and Verduyn Lunel (1993) and Hale and Verduyn Lunel (2002). For systems with up to three independent delays, a procedure is derived that can check strong stability in finite steps.

As shown in Gu (2012), the strong stability problem of such difference equation is closely related to the structured singular value problem (Doyle, 1982; Doyle, Wall, \& Stein, 1982; Packard \& Doyle, 1993; Zhou, Doyle, \& Glover, 1996). Therefore, the procedure derived here means that we have obtained a method to calculate the
structured singular value for up to three scalar complex uncertain blocks.

The remaining parts of this article is organized as follows. Section 2 discusses the relationship between the systems described by (1) and the ODPSC model described by (7). Section 3 develops the general theory of strong stability of system (1). These two sections are very similar to the contents of $\mathrm{Ma}, \mathrm{Gu}$, and Choubedar (2017). Section 4 presents a method to check strong stability of the system (1) in finite steps when there are not more than three independent parameters. Section 5 discusses the relationship between the strong stability problem and the structured singular value problem. Section 6 provides some numerical examples to illustrate the developed method.

## 2. Relations with ODPSC model

From the discussion above, we know that the characteristic quasipolynomial of the ODPSC form of difference equation set (6) has the form of (1). However, as will be shown in Theorem 1 below, for a given quasipolynomial $\Delta(s)$ of the form (1) with $K \geq 3$, it is not always possible to find an ODPSC difference equation set (6) such that its characteristic function $\Delta_{1}(s)$ is equal to $\Delta(s)$. Therefore, it is of independent interest to study the system (1).

Theorem 1. For a given quasipolynomial $\Delta(s)$ in the form of (1) with $K=3$, there exists a $3 \times 3$ matrix $D$ such that $\Delta_{1}(s)$ given in (7) satisfies $\Delta_{1}(s)=\Delta(s)$ if and only if the following inequality holds:

$$
\begin{align*}
& \left(a_{12} a_{3}+a_{13} a_{2}+a_{23} a_{1}-2 a_{1} a_{2} a_{3}-a_{123}\right)^{2} \\
\geq & 4\left(a_{1} a_{2}-a_{12}\right)\left(a_{2} a_{3}-a_{23}\right)\left(a_{3} a_{1}-a_{13}\right) . \tag{11}
\end{align*}
$$

The above theorem can be found as Theorem 1 in Ma et al. (2017) with complete proof.

## 3. Stability conditions

The strong stability condition of (6) can be found in Hale and Verduyn Lunel (1993) and Hale \& Verduyn Lunel (2002) with appropriate adaption described in Gu (2012). Here we will study the strong stability of the system (1). For complex numbers $\delta_{j}, j=$ $1, \ldots, K$, we allow a slight abuse of notation and write

$$
\begin{aligned}
& \Delta\left(\delta_{1}, \ldots, \delta_{K}\right) \\
= & 1+\sum_{m=1}^{K} \sum_{1 \leq i_{1}<i_{2}<\cdots<i_{m} \leq K} a_{i_{1} i_{2} \ldots i_{m}} \delta_{i_{1}} \delta_{i_{2}} \ldots \delta_{i_{m}} .
\end{aligned}
$$

Then
$\Delta\left(e^{-\tau_{1} s}, e^{-\tau_{2} s}, \ldots, e^{-\tau_{K} s}\right)=\Delta(s)$.
Theorem 2. The following statements are equivalent:
(i) System (1) is exponentially stable for a given set of rationally independent parameters $\tau_{1}>0, \tau_{2}>0, \ldots, \tau_{K}>0$.
(ii) For given nominal parameters $\tau_{1}^{0}>0, \tau_{2}^{0}>0, \ldots, \tau_{K}^{0}>0$, and an arbitrarily small $\varepsilon>0$, system (1) is exponentially stable for all positive parameters $\tau_{1}, \tau_{2}, \ldots, \tau_{K}$ that satisfy
$\left|\tau_{j}-\tau_{j}^{0}\right|<\varepsilon, j=1,2, \ldots, K$.
(iii) System (1) is exponentially stable for arbitrary positive parameters $\tau_{1}>0, \tau_{2}>0, \ldots, \tau_{K}>0$.
(iv)
$0 \notin\left\{\Delta\left(\delta_{1}, \delta_{2}, \ldots, \delta_{K}\right)\left|\left|\delta_{j}\right| \leq 1, j=1,2, \ldots, K\right\}\right.$.
(v)
$\min \left\{\Delta\left(\delta_{1}, \delta_{2}, \ldots, \delta_{K}\right)\left|\Delta \in \mathbb{R},\left|\delta_{j}\right|=1, j=1,2, \ldots, K\right\}\right.$
$>0$.

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[^0]:    This work is partially supported by National Science Foundation of China under Grant 61403199, the Natural Science Foundation of Jiangsu Province under Grant BK20140770, and the Fundamental Research Funds for the Central Universities of China under Grant 30916015105 . The material in this paper was partially presented at the 20th World Congress of the International Federation of Automatic Control, July 9-14, 2017, Toulouse, France. This paper was recommended for publication in revised form by Associate Editor Akira Kojima under the direction of Editor Ian R. Petersen.

    E-mail addresses: qma@njust.edu.cn (Q. Ma), kgu@siue.edu (K. Gu), nchoubedar@yahoo.co.uk (N. Choubedar).

