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Brief paper

Moment-based analysis of stochastic hybrid systems with renewal transitions[☆]

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ABSTRACT

Stochastic Hybrid Systems (SHS) constitute an important class of mathematical models that integrate discrete stochastic events with continuous dynamics. The time evolution of statistical moments is generally not closed for SHS, in the sense that the time derivative of the lower-order moments depends on higher-order moments. Here, we identify an important class of SHS where moment dynamics is automatically closed, and hence moments can be computed exactly by solving a system of coupled differential equations. This class is referred to as linear time-triggered SHS (TTSHS), where the state evolves according to a linear dynamical system. Stochastic events occur at discrete times and the intervals between them are independent random variables that follow a general class of probability distributions. Moreover, whenever the event occurs, the state of the SHS changes randomly based on a probability distribution. Our approach relies on embedding a Markov chain based on phase-type processes to model timing of events, and showing that the resulting system has closed moment dynamics. Interestingly, we identify a subclass of linear TTSHS, where the first and second-order moments depend only on the mean time interval between events, and invariant of higher-order statistics of event timing. TTSHS are used to model examples drawn from cell biology and nanosensors, providing novel insights into how noise is regulated in these systems.

1. Introduction

Stochastic hybrid systems (SHS) that combine continuous and discrete interactions are increasingly being used to model noise and uncertainty in physical, biological, and engineering systems. Specific applications include communication networks (Bohacek, Hespanha, Lee, & Obraczka, 2003; Hespanha, 2004, 2005a; Hu, 2006), network control systems (Antunes, Hespanha, & Silvestre, 2013b; Hespanha, 2014), air traffic control (Prandini & Hu, 2009; Visintini, Glover, Lygeros, & Maciejowski, 2006), biological systems (Antunes & Singh, 2014; Bortolussi & Policriti, 2008; Daigle, Soltani, Petzold, & Singh, 2015; Hu, Lygeros, & Sastry, 2004; Singh & Hespanha, 2010; Soltani & Singh, 2016; Soltani, Vargas-Garcia, Antunes, & Singh, 2016; Vargas-García, Soltani, & Singh, 2016), power grids (Dhople, Chen, DeVille, & Dominguez-Garcia, 2013; Wang & Crow, 2011), modeling of energy grids and smart buildings (David, Du, Larsen, Mikučionis, & Skou, 2012; Strelec, Macek, & Abate, 2012). Interested readers are referred to recent reviews for an introduction to SHS (Hespanha, 2006; Hu, Lygeros, & Sastry, 2000; Teel, Subbaraman, & Sferlazza, 2014).

Traditional analysis of SHS relies heavily on various Monte Carlo simulation techniques, which come at a significant computational cost (Gillespie, 1976; Gillespie & Petzold, 2003). Since one is often interested in computing only the lower-order moments of the state variables, much time and effort can be saved by directly computing these statistical moments without having to run Monte Carlo simulations. Unfortunately, moment calculations in SHS can be nontrivial due to the problem of unclosed dynamics: the time evolution of lower-order moments of the state space depends on higher-order moments (Singh & Hespanha, 2005). In such cases, moments are solved by employing closure schemes, that close the system of differential equations by approximating higher-order moments as nonlinear functions of lower-order moments (Gillespie, 2009; Lee, Kim, & Kim, 2009; Singh & Hespanha, 2006, 2011; Soltani, Vargas-Garcia, & Singh, 2015).

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The problem of moment closure leads to an interesting question: are there classes of SHS where moments can be computed exactly without the need for closure techniques? Here, we identify such a class of SHS known as time-triggered SHS (TTSHS) that are a special case of piecewise-deterministic Markov processes (Costa & Dufour, 2008; Davis, 1993). The main ingredients of TTSHS are as follows:

(1) A continuous state $\mathbf{x}(t) \in \mathbb{R}^n$ that evolves according to a stable linear dynamical system

$$\dot{\mathbf{x}}(t) = \hat{a} + A\mathbf{x}(t), \tag{1}$$

for some constant vector $\hat{a}$ and matrix A. While previous studies considered continuous dynamics of the form (1), we extend our analysis to include TTSHS with stochastic differential equations.

(2) Stochastic events occur at discrete times $\mathbf{t}_s$, $s \in \{1, 2, \ldots\}$, and the intervals $\mathbf{t}_s - \mathbf{t}_{s-1}$ are independent and identical random variables drawn from a given probability density function. These events can be referred to as renewal transitions, as their timing is determined by an underlying renewal process.

(3) A reset map defines the change in $\mathbf{x}$ when the event occurs

$$\mathbf{x}(\mathbf{t}_s) \mapsto \mathbf{x}_+(\mathbf{t}_s), \tag{2}$$

where $\mathbf{x}_+(\mathbf{t}_s)$ denotes the state of the system just after the event. While prior work has considered a deterministic linear reset map

$$\mathbf{x}(\mathbf{t}_s) \mapsto J\mathbf{x}(\mathbf{t}_s) \tag{3}$$

(Antunes, Hespanha, & Silvestre, 2010, 2012, 2013a), we allow for both state-dependent and state-independent noise sources in $\mathbf{x}_+(\mathbf{t}_s)$.

Our goal is to connect moments of the continuous state to the statistics of the time interval $\mathbf{T} \equiv \mathbf{t}_s - \mathbf{t}_{s-1}$. The key contribution of this work is to model arrival of events using phase-type processes (Tijms, 1994), and show that the resulting systems has closed moment dynamics. More specifically, the time derivative of an appropriately selected vector of moments depends only on itself, and not on higher-order moments. As a consequence, moments can be computed exactly by solving a system of differential equations. For the sake of simplicity, we focus on computing the first and second-order moments, but the ideas can be generalized to obtain any higher-order moment. In addition, a subclass of TTSHS is identified, where the first and second-order moments of $\mathbf{x}$ depend only on the mean time interval between events. In this case, making the arrival of events more random (for fixed mean arrival times) will not result in higher noise in $\mathbf{x}$. These methods are illustrated on examples drawn from different disciplines.

2. TTSHS model formulation

Let the continuous state $\mathbf{x}(t) \in \mathbb{R}^n$ evolves according to a set of stochastic differential equations as

$$d\mathbf{x}(t) = (\hat{a} + A\mathbf{x}(t))dt + (G + B\mathbf{x}(t)\mathbb{1}_n)d\boldsymbol{w}_t, \tag{4}$$

where $G \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are constant matrices, $\mathbb{1}_n$ is a $1 \times n$ unit matrix. Further $\boldsymbol{w}_t$ is a n-dimensional Weiner process satisfying

$$\langle d\boldsymbol{w}_t\rangle = 0, \quad \langle d\boldsymbol{w}_t d\boldsymbol{w}_t^\top\rangle = I_n dt, \tag{5}$$

where I_n is a $n \times n$ Identity matrix, and the symbol $\langle\ \rangle$ denotes the expected value.

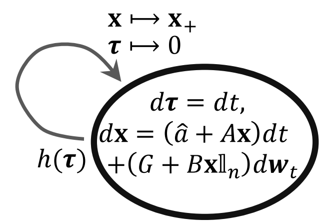

Fig. 1. Schematic of a linear time-triggered stochastic hybrid system. The state evolves according to a set of stochastic differential equation and events occur randomly with hazard rate $h(\boldsymbol{\tau})$, where the timer $\boldsymbol{\tau}$ measures the time since the last event. Choosing the hazard rate as (6), ensures that the time between events follows a continuous probability density function f. Whenever the event occurs the timer is set to zero and $\mathbf{x}$ changes via (7).

The timing of events in TTSHS can be modeled through a timer $\boldsymbol{\tau}$, that measures the time elapsed since the last event. This timer is reset to zero whenever an event occurs and increases over time as $d\boldsymbol{\tau} = dt$ in between events. Let the time intervals between events follow a continuous positively-valued probability density function f. Then, the transition intensity for the event is given by the hazard function

$$h(\tau) \equiv \frac{f(\tau)}{1 - \int_{y=0}^{\tau} f(y)dy} \tag{6}$$

(Evans, Hastings, & Peacock, 2000; Ross, 2010). In particular, the probability that an event occurs in the next infinitesimal time interval $(t, t + dt]$ is $h(\boldsymbol{\tau})dt$. This formulation of the event arrival process via a timer allows representation of TTSHS as a state-driven SHS (Fig. 1). Hence, existing tools such as, Kolmogorov equations and Dynkin's formulas for obtaining time evolution of moments can be employed for studying stochastic dynamics of TTSHS (Hespanha, 2014). Having defined the timing of events, we next focus on how the events alter the state of the system.

Whenever an event occurs, $\mathbf{x}$ is reset to $\mathbf{x}_+$, where $\mathbf{x}_+$ is a random variable with following statistics

$$\langle \mathbf{x}_+(\mathbf{t}_s)\rangle = J\mathbf{x}(\mathbf{t}_s) + R, \tag{7a}$$

$$\begin{aligned}&\left\langle \mathbf{x}_+(\mathbf{t}_s)\mathbf{x}_+^\top(\mathbf{t}_s)\right\rangle - \langle \mathbf{x}_+(\mathbf{t}_s)\rangle\langle \mathbf{x}_+(\mathbf{t}_s)\rangle^\top \\ &\quad = Q\mathbf{x}(\mathbf{t}_s)\mathbf{x}^\top(\mathbf{t}_s)Q^\top + C\mathbf{x}(\mathbf{t}_s)D^\top + D\mathbf{x}^\top(\mathbf{t}_s)C^\top + E,\end{aligned} \tag{7b}$$

here $J \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times 1}$, $Q \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times 1}$, and $E \in \mathbb{R}^{n\times n}$ are constant matrices. Further E is symmetric. Note that the mean of $\mathbf{x}_+$ is a linear affine function of $\mathbf{x}$, which is a generalization over the linear map (3) previously used. The covariation matrix of $\mathbf{x}_+$ is defined by (7b) and covers a wide range of possibilities. For example, $Q = C = D = E = 0$ imply $\mathbf{x}_+ = J\mathbf{x} + R$ with probability one. Moreover, non-zero Q, C, D, and E can be used to incorporate constant or state-dependent noise in $\mathbf{x}_+$. In the following sections, we show how statistical moments of $\mathbf{x}(t)$ can be computed exactly for TTSHS illustrated in Fig. 1. We first consider a subclass of TTSHS, where events only impart noise to the system, in the sense that the average value of $\mathbf{x}$ just after the event is the same as its value just before the event.

3. Moment dynamics of TTSHS with noise-imparting events

Consider a subclass of the TTSHS with $J = I$, $R = Q = 0$ reducing (7) to

$$\langle \mathbf{x}_+(\mathbf{t}_s)\rangle = \mathbf{x}(\mathbf{t}_s), \tag{8a}$$

$$\begin{aligned}&\left\langle \mathbf{x}_+(\mathbf{t}_s)\mathbf{x}_+^\top(\mathbf{t}_s)\right\rangle - \langle \mathbf{x}_+(\mathbf{t}_s)\rangle\langle \mathbf{x}_+(\mathbf{t}_s)\rangle^\top \\ &\quad = C\mathbf{x}(\mathbf{t}_s)D^\top + D\mathbf{x}^\top(\mathbf{t}_s)C^\top + E.\end{aligned} \tag{8b}$$