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Brief paper

On the consensus of homogeneous multi-agent systems with arbitrarily switching topology☆

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ABSTRACT

In this paper we investigate the consensus problem under arbitrary switching for homogeneous multi-agent systems with switching communication topology, by assuming that each agent is described by a single-input stabilizable state–space model and that the communication graph is connected at every time instant. Under these assumptions, we construct a common quadratic positive definite Lyapunov function for the switched system describing the evolution of the disagreement vector, thus showing that the agents always reach consensus. In addition, the proof leads to the explicit construction of a constant state-feedback matrix that allows the multi-agent system to achieve consensus.

1. Introduction

Research efforts on multi-agent systems, in general, and consensus problems, in particular, have been quite impressive in the last decade. Originated by some remarkable contributions that still represent the reference points of any paper on the subject (Fax & Murray, 2004; Olfati-Saber, Fax, & Murray, 2007; Ren & Atkins, 2007; Ren & Beard, 2005; Tsitsiklis, 1984), the research flourished by addressing increasingly more complex set-ups, and hence taking into account higher order (possibly nonlinear) models for the agents, time-varying communication topologies, antagonistic interactions, communication delays, output feedback, packet-loss, etc. (see, e.g., Li, Chen, Su, & C., 2015; Li, Duan, & Lewis, 2014; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Scardovi & Sepulchre, 2009; Su, & Lin, 2016; Xia, Cao, & Johansson, 2016).
Even if a significant portion of the research in this area focuses on first and second order systems (Meng, Shi, Johansson, & Hong; Xia et al., 2016), a good number of contributions have investigated the case when the agents' dynamics is described by a generic state–space model. While in the early contributions the communication topology was supposed to be fixed (Wieland, Kim, & Allgower, 2011; Wieland, Kim, Scheu, & Allgower, 2008), more recent papers have explored the case of a time-varying communication topology, possibly switching among a finite set of configurations (Su & Huang, 2012; Wang, Cheng, & Hu, 2008; Wen, Duan, Chen, & Yu, 2014; Wen & Ugrinovskii, 2014).

All the literature addressing the case of higher order agents with switching communication topologies has been able to relax the connectedness constraint on every communication graph, at the price of imposing additional constraints not only on the switching signal but also on the agents' model. Specifically, in Su and Huang (2012) and Wu, Qin, Yu, and Allgower (2013) the consensus problem is solved and an explicit solution is provided, by assuming that the agent's state model is stabilizable, the state matrix A is simply stable, the switching signal describing how the communication topology varies has a minimum dwell-time and ensures that the time-varying communication graph $\mathcal{G}(t)$ is uniformly connected over $[0, +\infty)$. In Qin and Yu (2014) the state matrix A satisfies some algebraic constraint, the input to state matrix B is of full row rank (a sufficient condition for controllability), switching signals have a dwell time, and the communication topologies are repeatedly jointly rooted. Under these conditions, the consensus problem is solvable and a state feedback matrix is explicitly constructed. In Wang et al. (2008), consensus has been investigated, under the assumption that agents are controllable and switching signals have a dwell-time, both in case the communication network over which agents communicated is connected at every time and in case it is frequently connected with a certain period T.

In most of these contributions, dwell time and controllability have been fundamental requirements in order to design a state

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feedback controller that ensures a sufficiently rapid convergence. For instance, the proof of Theorem 1 in Wang et al. (2008) heavily relies on the possibility to freely allocate the eigenvalues of the matrices $A - \lambda_i BK$, involved in the disagreement dynamics, and on the existence of a dwell time. It is interesting to understand under what conditions multi-agent consensus can be guaranteed corresponding to *every* switching signal and not only corresponding to switching signals with dwell-time.

It is a standard result for switched systems that if all the subsystems are asymptotically stable, then a dwell-time can always be found such that the switched system is asymptotically and hence exponentially stable. However, asymptotic stability of the subsystems alone does not ensure asymptotic stability of the switched system for every switching signal.

In this paper we investigate the consensus problem under arbitrary switching. This strong requirement on the system performances necessarily imposes that the communication network is connected at every time. On the other hand, it turns out that the stabilizability of the agents' dynamics is necessary and sufficient for the problem solvability, just like it happens when we consider a fixed connected communication network (Wieland et al., 2011). Even more, we provide an explicit solution to the consensus problem. The paper set-up is inspired by the one adopted in Su and Huang (2012), but we extend the agents' model decomposition adopted in the previously mentioned reference to the case when the state matrix has also eigenvalues with positive real part. Subsequently, we construct a quadratic positive definite function that ensures the asymptotic stability of the switched system describing the dynamics of the disagreement vector, and thus prove consensus. It is worth remarking that, in general, it is hard or even impossible to construct a common quadratic Lyapunov function for the consensus error system of a multi-agent system with switching topology. When so (see e.g. Wen et al., 2014; Wen, Hu, Yu, & Chen, 2014; Wen & Ugrinovskii, 2014), multiple Lyapunov functions have been proposed to stabilize or verify the stability.

It is worth comparing our results with those derived in Li, Ren, Liu, and Xie (2013), where a consensus protocol for homogeneous multi-agent systems with arbitrarily switching topologies is proposed, by assuming that the communication graph is connected (and undirected) at every time instant. The set-up adopted in Li et al. (2013) is rather different from the one considered in this paper, since the switching takes place among all possible undirected, connected and *unweighted* communication graphs, but the weights attributed to the graph edges are regarded as control variables that continuously update. So, the switching is not among a finite number of undirected, connected and *weighted* communication graphs, but weights can be adaptively modified. The advantage of this adaptive consensus protocol is that it can be implemented in a completely distributed way by the agents. The con is that the controller significantly increases in size. Indeed, if n is the size of the agents' state and N is the number of the agents, the overall controlled system in Li et al. (2013) has size $2nN + N(N-1)/2$, since the adaptive controller updates both a "protocol state" of size n for each agent, and the distinct $N(N-1)/2$ weights of the edges of the undirected graph at every time instant. In this paper, we will use a static controller and the overall controlled system will have size nN.

The paper is organized as follows: in Section 2 we present some background material on matrices, graphs, and Laplacians. Section 3 presents the problem set-up. In Section 4 some preliminary analysis is carried on that allows to simplify the set-up and to reduce the consensus problem to a stabilization problem for a lower-order switched system with autonomous subsystems. A constructive proof of the main result, stating that if the communication network is connected at every time and the agents' model is stabilizable, then consensus can always be achieved, is given in Section 5, together with a simple algorithm to explicitly construct a state feedback matrix ensuring consensus.

2. Background material

If p is a positive integer, we denote by $[1, p]$ the finite set $\{1, 2, \dots, p\}$. $\mathbf{e}_i$ is the ith *canonical vector* in $\mathbb{R}^N$, where N is always clear from the context. $\mathbf{1}_N$ and $\mathbf{0}_N$ are the N-dimensional vectors with all entries equal to 1 and to 0, respectively. Given $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ the spectrum of A and by $\lambda_{\max}(A) \in \mathbb{R}$ its *spectral abscissa*, defined as $\lambda_{\max}(A) := \max\{\Re(\lambda), \lambda \in \sigma(A)\}$. A is *Hurwitz* if $\lambda_{\max}(A) < 0$. The *Kronecker (or tensor) product* of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is

$$C = [A \otimes B] := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{pm\times qn}.$$

An $n \times n$ matrix A, $n > 1$, is *reducible* if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*.

$\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix A with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A \geq 0$); if $A \geq 0$ and at least one entry is positive, A is a *positive matrix* ($A > 0$).

A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. For a Metzler matrix, the spectral abscissa is always an eigenvalue (namely the eigenvalue with maximal real part is always real). Given two Metzler matrices A and $\bar{A} \in \mathbb{R}^{n\times n}$, the following monotonicity property holds (Son & Hinrichsen, 1996): if $A \leq \bar{A}$, then $\lambda_{\max}(A) \leq \lambda_{\max}(\bar{A})$; if in addition $\bar{A}$ is irreducible, then $A < \bar{A}$ implies $\lambda_{\max}(A) < \lambda_{\max}(\bar{A})$.

An *undirected, weighted graph* is a triple (Mohar, 1991) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs, and $\mathcal{A} = \mathcal{A}^\top \in \mathbb{R}_+^{N\times N}$ is the (positive and symmetric) *adjacency matrix* of the weighted graph $\mathcal{G}$. In this paper we assume that $\mathcal{G}$ has no self-loops, namely each diagonal entry $[\mathcal{A}]_{ii}$, $i \in [1, N]$, is zero. A sequence $j_1 \leftrightarrow j_2 \leftrightarrow j_3 \leftrightarrow \cdots \leftrightarrow j_k \leftrightarrow j_{k+1}$ is a *path* of length k connecting j_1 and j_{k+1} provided that $(j_1, j_2), (j_2, j_3) \dots, (j_k, j_{k+1}) \in \mathcal{E}$. A graph is said to be *connected* if for every pair of distinct vertices $i, j \in \mathcal{V}$ there is a path connecting i and j. This is equivalent to the fact that $\mathcal{A}$ is an irreducible matrix.

We define the *Laplacian matrix* $\mathcal{L} \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ as $\mathcal{L} := \mathcal{C} - \mathcal{A}$, where $\mathcal{C} \in \mathbb{R}_+^{N\times N}$ is a diagonal matrix whose ith diagonal entry is the *weighted degree* of vertex i, i.e. $[\mathcal{C}]_{ii} := \sum_{l=1}^{N} [\mathcal{A}]_{il}$. Accordingly, the Laplacian matrix $\mathcal{L} = \mathcal{L}^\top$ takes the following form:

$$\mathcal{L} = \begin{bmatrix} \ell_{11} & \ell_{12} & \dots & \ell_{1N} \\ \ell_{12} & \ell_{22} & \dots & \ell_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1N} & \ell_{2N} & \dots & \ell_{NN} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{N} [\mathcal{A}]_{1j} & -[\mathcal{A}]_{12} & \dots & -[\mathcal{A}]_{1N} \\ -[\mathcal{A}]_{12} & \sum_{j=1}^{N} [\mathcal{A}]_{2j} & \dots & -[\mathcal{A}]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathcal{A}]_{1N} & -[\mathcal{A}]_{2N} & \dots & \sum_{j=1}^{N} [\mathcal{A}]_{Nj} \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

As all rows of $\mathcal{L}$ sum up to 0, $\mathbf{1}_N$ is always a right eigenvector of $\mathcal{L}$ corresponding to the eigenvalue 0. The following lemma