Automatica 84 (2017) 174–180

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A bounded real lemma type-result with respect to the anisotropic norm setup for stochastic systems with multiplicative noise ☆

Adrian-Mihail Stoica [a], Isaac Yaesh [b]

[a] University "Politehnica" of Bucharest, Faculty of Aerospace Engineering, Str. Polizu, No. 1, 011063, Bucharest, Romania
[b] Control Department, IMI Advanced Systems Div., P.O.B. 1044/77, Ramat Hasharon, 47100, Israel

ARTICLE INFO

Article history:
Received 30 March 2015
Received in revised form 9 March 2017
Accepted 3 July 2017
Available online 17 August 2017

Keywords:
Discrete-time systems
Stochastic systems
Multiplicative noise
Norms
Riccati equations

ABSTRACT

The anisotropic norm of discrete-time linear stochastic systems with state dependent noise is considered. Using first principles analysis applying completing the square arguments, it is proved that the anisotropic norm of such systems is upper bounded by a given positive real scalar, if a specific Riccati equation has a stabilizing positive semidefinite solution satisfying two additional conditions. It is shown that these conditions are sufficient and necessary for the boundedness of the anisotropic norm. Numerical algorithms to determine the stabilizing solution of this Riccati equation allowing thus to compute the anisotropic norm of stochastic systems with multiplicative noise are also presented. The theoretical results are illustrated by numerical examples.

1. Introduction

The problems of robust optimal control and filtering received much attention over the last seven decades. Early solutions for these problems were presented by Kwakernaak and Sivan (1987) and robustness issues due to modelling errors were considered in e.g. Stein and Athans (1992). Structured modelling errors affecting the state–space model can be assumed to be either deterministic (e.g. Xie, Fu, and deSouza (1992) and Young, Newlin, and Doyle (1991)) or stochastic (Costa & Kubrusly, 1996). In the latter case, the system state–space matrices are subject to additive random perturbations of white noise type, leading to systems involving state-multiplicative white noise. This type of stochastic systems have been studied over the last few decades (see, for instance, Wonham (1970), Dragan, Morozan, and Stoica (2010) and Gershon, Shaked, and Yaesh (2005)). Real-life systems differ from their nominal models also due to exogenous signals serving either as a process noise (i.e. the input to the systems) or as a measurement noise. When the exogenous signals are of white noise type, then H_2− norm minimization is applied, leading to the Kalman filter (Kalman, 1960) and Linear Quadratic Gaussian (LQG) control. An alternative modelling of the exogenous inputs is based on deterministic bounded energy signals. Such formulations lead to the H_∞-norm based framework (Zames, 1981) and are applied in both filtering (Grimble, 0000; Simon, 2006) and control (Yaesh & Shaked, 1991). Many practical applications, however, require a compromise between the H_2 and the H_∞-norm minimization since the latter may not be suitable when the considered signals are strongly coloured (e.g. periodic signals). On the other hand, H_∞-optimization may poorly perform when these signals are weakly coloured (e.g. white noise). For such cases mixed H_2/H_∞ norm minimization problems have been formulated and analysed (see, e.g. Bernstein and Haddad (1989), Dragan et al. (2010), Rotstein & Sznaier (1998) and Zhang, Huang, and Xie (2008)). A promising alternative to accomplish such compromise is to use the so-called *a-anisotropic norm* (Vladimirov, Kurdyukov, & Semyonov, 1995, 1996) since it offers and intermediate topology between the H_2 and H_∞ norms. More precisely, if the coloured signal is generated by an m-dimensional exogenous input, the a-anisotropic norm $|||F|||_a$ of a stable system F has the property (see, for instance Vladimirov et al. (1996)):

$$\frac{1}{\sqrt{m}}\|F\|_2 = |||F|||_0 \le |||F|||_a \le \|F\|_\infty = \lim_{a\to\infty} |||F|||_a$$

where $|||F|||_a$ is defined by

$$|||F|||_a = \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{1}$$

$\mathcal{G}_a$ denotes the set of all stochastic systems of form (4) with the *mean anisotropy* $\bar{A}(G) \le a$. The mean anisotropy of a stationary Gaussian sequence was introduced in Vladimirov et al. (1995)

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Valery Ugrinovskii under the direction of Editor Ian R. Petersen.

E-mail addresses: adrian.stoica@upb.ro (A.-M. Stoica), iyaesh@imi-israel.com (I. Yaesh).

http://dx.doi.org/10.1016/j.automatica.2017.07.023

and represents an entropy theoretic measure of the deviation of finite-dimensional probability distributions of such a sequence from the corresponding isotropic distributions of a Gaussian white noise sequence with zero mean and a scalar covariance matrix. In Vladimirov et al. (1996), it is proved based on the Szegö–Kolmogorov theorem (Rozanov, 1990), that the mean anisotropy of a signal $w(k)$, $-\infty < k < \infty$, generated by an m-dimensional Gaussian white noise with zero mean and identity covariance applied to a stable linear system G with m outputs, has the form

$$\bar{A}(G) = -\frac{1}{2} \ln \det \left(\frac{mE\left[\tilde{w}(0)\tilde{w}(0)^T\right]}{E\left[|w(0)|^2\right]} \right), \tag{2}$$

where $E[\tilde{w}(0)\tilde{w}(0)^T]$ is the covariance of the prediction error $\tilde{w}(0) := w(0) - E[w(0)|w(k), k < 0]$. In the case when the output w of the filter G is a zero mean Gaussian white noise (i.e. its optimal estimate is just zero), $w(0)$ cannot be estimated from its past values and $\tilde{w}(0) = w(0)$ which leads to $\bar{A}(G) = 0$. In Kurdyukov, Maksimov, and Tchaikovsky (2010), conditions for the anisotropic norm boundedness are given in terms of a non convex optimization problem while in Tchaikovsky and Kurdyukov (2011) a convex form of the Bounded Real Lemma (BRL) type result with respect to the anisotropic norm was obtained, in both papers linear discrete-time systems being considered. One of the leading motivations to use the anisotropic norm is the fact $\|F\|_a \leq \|F\|_\infty$ making it a relaxed version of the H_∞-norm for many practical cases in which the driving noise signals can be characterized not just by their finite energy, but as outputs of a colouring linear systems in a certain class, where the colouring filters are of a finite anisotropy. In a case study presented in Tchaikovsky (2012) it is shown that for a TU-154 type aircraft landing system, the H_∞ controller is more efficient than the corresponding H_2 controller for a wind shear profile (which is a coloured rather than a white noise process) but, as could be expected, is more conservative, in the sense of higher gains and subsequently larger control actions; moreover, the anisotropic-norm based controller (based on an appropriate anisotropic norm bound) is less conservative than the H_∞ controller and requires significantly smaller control actions.

The aim of the present paper is to derive convex characterization of BRL type conditions for the bound on the anisotropic norm of *stochastic systems with multiplicative noise*. All developments of the present paper utilize time domain representations of the signals and direct calculations. The obtained results provide a generalization, for the case of systems with randomly perturbed state–space matrices considered in Stoica and Yaesh (2012). Note that when G is a linear system without multiplicative noise, then its output w has a Gaussian distribution. When, however, G is corrupted with multiplicative noise as considered in this paper, the equivalent definition given above for $\bar{A}(G)$ no longer holds, and the higher moments than just the spectral density are involved. In spite of this fact, in the present paper, we adopt an anisotropic-norm setup, where the simple definition of (2) in terms of second order moments only of $w(0)$ and its estimate. We will see in the sequel, that this definition leads to the result of our Theorem 1, which is consistent both with the anisotropic norm-related results of Kurdyukov et al. (2010) and Tchaikovsky and Kurdyukov (2011) for linear systems without multiplicative noise and with the H_∞-norm related results of Dragan et al. (2010) for systems with multiplicative noise.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}$ denotes the set of real numbers whereas $\mathcal{Z}_+$ stands for the non-negative integers. Moreover, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$ ($P \geq 0$), for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite (positive semi-definite). The trace of a matrix Z is denoted by $Tr(Z)$, and $|v|$ denotes the Euclidean norm of the vector v. Finally note that the terms Lyapunov and Riccati equations in this paper, refer to generalized versions of the standard equations appearing in the H_2 and H_∞ control literature.

2. Preliminaries and problem statement

Consider the stochastic system with multiplicative noise, F, described by

$$\begin{aligned} x(t+1) &= \mathcal{A}(t)x(t) + \mathcal{B}(t)w(t) \\ y(t) &= Cx(t) + Dw(t),\ t = 0, 1, \ldots \end{aligned} \tag{3}$$

where the randomly perturbed state–space matrices are given by

$$\mathcal{A}(t) := A_0 + \sum_{i=1}^{r} \xi_i(t)A_i$$

$$\mathcal{B}(t) := B_0 + \sum_{i=1}^{r} \xi_i(t)B_i$$

and where $\xi(t) = (\xi_1(t), \ldots, \xi_r(t))^T : \Omega \to \mathcal{R}^r$ are independent random vectors on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ numbered by nonnegative integers $t = 0, 1, \ldots$. It is assumed that $\{\xi(t)\}_{t\geq 0}$ satisfies the conditions $E[\xi(t)] = 0$ and $E\left[\xi(t)\xi^T(t)\right] = I_r$, $t = 0, 1, \ldots, E[\cdot]$ denoting the expectation of the random variable $(\cdot)$. The matrices of the state space model (3) have the dimensions as follows: $A_i \in \mathcal{R}^{n\times n}$, $B_i \in \mathcal{R}^{n\times m}$, $i = 0, 1 \ldots, r$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{p\times m}$.

Definition 1. A stochastic system with multiplicative noise of form (3) with $B_i = 0$, $i = 0, 1, \ldots, r$ is called exponentially stable in mean square (ESMS) if there exist $\beta \geq 1$ and $\rho \in (0, 1)$ such that $E\left[|\Phi(t,s)x(s)|^2\right] \leq \beta\rho^{t-s}E\left[|x(s)|^2\right]$ for all $t \geq s \geq 0$, $x(s) \in \mathcal{R}^n$, where $\Phi(t, s)$ denotes the fundamental matrix solution of (3).

Definition 2. The H_2-type norm of the ESMS system (3) is defined as

$$\|F\|_2 = \left[\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{t=0}^{\ell} E\left[y^T(t)y(t)\right] \right]^{\frac{1}{2}},$$

where $\{y(t)\}_{t\in\mathcal{Z}_+}$ is the output of the system (3) with zero initial conditions generated by the sequence $\{w(t)\}_{t\in\mathcal{Z}_+}$ of independent random vectors with the property that $E[w(t)] = 0$ and $E\left[w(t)w^T(t)\right] = I_m$, $\{w(t)\}_{t\in\mathcal{Z}_+}$ being assumed independent of the stochastic process $\{\xi(t)\}_{t\in\mathcal{Z}_+}$.

The next result provides a method to compute the H_2 norm of the stochastic system (3) (see e.g. Dragan et al. (2010)).

Lemma 1. *The H_2 type norm of the ESMS system (3) is given by $\|F\|_2 = \left(Tr\left(\sum_{i=0}^{r} B_i^T XB_i + D^T D\right)\right)^{\frac{1}{2}}$ where $X \geq 0$ is the solution of the generalized Lyapunov equation $X = \sum_{i=0}^{r} A_i^T XA_i + C^T C$.*

The dual result is given by the following lemma (Dragan et al., 2010).

Lemma 2. *The H_2 type norm of the ESMS system (3) is also given by $\|F\|_2 = \left(Tr\left(CYC^T + DD^T\right)\right)^{\frac{1}{2}}$ where $Y \geq 0$ is the solution of the generalized Lyapunov equation $Y = \sum_{i=0}^{r} A_iYA_i^T + \sum_{i=0}^{r} B_iB_i^T$.*

Let $L^2(\mathcal{Z}_+ \times \Omega, \mathcal{R}^m)$ denote the space of all sequences $w = \{w(t)\}_{t\in\mathcal{Z}_+}$ of m-dimensional vectors with $\|w\|^2 := \sum_{t=0}^{\infty} E\left[|w(t)|^2\right] < \infty$ and $\tilde{L}^2(\mathcal{Z}_+ \times \Omega, \mathcal{R}^m)$ denote the space of all $w \in L^2(\mathcal{Z}_+ \times \Omega, \mathcal{R}^m)$ such that $w(t)$ is measurable with respect to $\mathcal{F}_t$ for every $t \in \mathcal{Z}_+$, $\mathcal{F}_t \subset \mathcal{F}$ denoting a family of σ-algebras generated by $\xi(s)$, $s < t$. In Morozan (1998), it is proved that if