Automatica 84 (2017) 199–204

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Robust input-to-output stabilization of nonlinear systems with a specified gain☆

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ARTICLE INFO

Article history:
Received 11 October 2015
Received in revised form 7 March 2017
Accepted 5 June 2017
Available online 17 August 2017

Keywords:
Nonlinear systems
Stabilization
Robust control
Synchronization
Multi-agent systems

ABSTRACT

Robust stabilization of nonlinear systems has been extensively studied subject to both static and dynamic uncertainties. In particular, a class of nonlinear controllers can be explicitly constructed based on recursively applying the small gain theorem. In this paper, we further investigate the so-called robust input-to-output stabilization problem for systems with additional external input (or perturbation, rather than the control input). It aims to explicitly construct a controller such that the system is input-to-output stable (IOS) from the external input to the specified output, moreover, with a specified IOS gain.

1. Introduction

We consider a class of nonlinear systems modeled by the following lower triangular dynamics

$$\begin{aligned}
\dot{z}_j &= q_j(\vec{z}_j, \vec{x}_j, \sigma, d)\\
\dot{x}_j &= f_j(\vec{z}_j, \vec{x}_j, \sigma, d) + b_j(d)x_{j+1},\ j = 1, \ldots, r\\
y &= x_1,
\end{aligned} \tag{1}$$

where $\vec{z}_j := \mathrm{col}(z_1, \ldots, z_j)$ with $z_j \in \mathbb{R}^{n_j}$ and $\vec{x}_j := \mathrm{col}(x_1, \ldots, x_j)$ with $x_j \in \mathbb{R}$ are the states, $u := x_{r+1}$ is the control input, $\sigma \in \mathbb{R}^{n_\sigma}$ represents the external input (or called perturbation), $d \in \mathbb{D}$ represents system uncertainties (unknown parameters and/or disturbances) in a known compact set $\mathbb{D}$, and r is the relative degree. Denote the full state as $\xi := \mathrm{col}(\vec{z}_r, \vec{x}_r)$. It is assumed that $\xi = 0$ is the equilibrium point of the zero-input unperturbed system ($u = 0$, $\sigma = 0$), that is, the functions q_j and f_j satisfy $q_j(0, 0, 0, d) = 0$ and $f_j(0, 0, 0, d) = 0$. It is also assumed that the functions are sufficiently smooth. In (1), the function q_i may not be known precisely and/or the state $\vec{z}_r$ cannot be used for feedback control, so the dynamics governing z_j are regarded as dynamic uncertainties. The vector d represents static uncertainties. In other words, we consider the scenario that only the state $\vec{x}_r$ is measurable and available for feedback design.

For systems free of external input σ, i.e., $n_\sigma = 0$, the global robust stabilization problem has been well investigated in literature under various characterizations of system complexity. The early backstepping technique was developed in, e.g., Byrnes and Isidori (1989) for systems involving neither dynamic uncertainty nor static uncertainty. For systems containing only static uncertainty, the problem was dealt with using robust and/or adaptive backstepping approaches in, e.g., Chen and Huang (2002) and Lin and Gong (2003). The technique becomes more complicated for systems that contain dynamic uncertainty whether it appears only at the top subsystem with $j = 1$ (Dashkovskiy, Pavlichkov, & Jiang, 2013; Jiang & Praly, 1998) or at all subsystems (Jiang & Marcels, 1997; Liu & Jiang, 2015). There are basically two approaches to handle the global robust stabilization problem in this scenario. One is based on the nonlinear small gain theorem in the context of input-to-state stability; see the original work in Jiang and Marcels (1997); Jiang, Teel, and Praly (1994). The concepts of input-to-state stability, input-to-output stability, output–input stability, etc., were originally proposed for stability analysis of nonlinear systems in, e.g., Liberzon, Morse, and Sontag (2002) and Sontag (1989). The small gain theorem based method was also used in Huang and Chen (2004) in dealing with the output regulation problem after being converted to a global robust stabilization problem. The other is

☆ This research was supported under Australian Research Council's Discovery Projects funding scheme (project number DP150103745) and opening fund of the State Key Laboratory of Digital Manufacturing Equipment and Technology, Huazhong University of Science and Technology, China (No. DMETKF2016019). The material in this paper was partially presented at the 12th World Congress on Intelligent Control and Automation, June 12–15, 2016, Guilin, China. This paper was recommended for publication in revised form by Associate Editor Hyungbo Shim under the direction of Editor Daniel Liberzon.

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http://dx.doi.org/10.1016/j.automatica.2017.07.019
0005-1098/

based on the Lyapunov direct approach; see, e.g., Chen and Huang (2004, 2005a). More extensive discussion and relevant references can be found in the textbook (Chen & Huang, 2015).

This paper studies a kind of input-to-output stabilization problem which is different from the traditional stabilization problem in the sense that a nontrivial external perturbation σ enters the system. This situation is motivated by the recent development on output synchronization of nonlinear heterogeneous multi-agent systems in, e.g., Chen, Chen, and Zhang (2014) and Zhu, Chen, and Middleton (2016). In particular, the input-to-output stabilization problem was solved in Chen et al. (2014) for the class of lower triangular systems, but without specifically calculating the resulting input-to-output gain. It was further investigated in Zhu et al. (2016) that a controller is able to render an input-to-output stable closed-loop system with a specified nonlinear gain function. It was found that the treatment for a specified gain is essentially challenging. Therefore, the specific controller design in Zhu et al. (2016) is only for systems of a unity relative degree, i.e., $r = 1$.

The direct objective of this paper is to solve the input-to-output stabilization problem for nonlinear lower triangular systems with a specified input-to-output gain, for the general case with $r > 1$. The main technical challenge is caused by existence of dynamic uncertainties and recursive development for systems of high relative degrees. Recall that the existing backstepping approaches based on nonlinear small gain theorem typically treat the $(z_1, x_1, \ldots, z_j, x_j)$-subsystem as one complete system and the (z_{j+1}, x_{j+1})-subsystem as another to form recursion. They do not offer a mechanism to separately handle the z_j and x_j-subsystems at each recursive step. It does not cause any issue until a (sufficiently small) input-to-output gain is required in the scenario studied in this paper. In particular, as the z_j-dynamics are regarded as dynamic uncertainties that cannot be directly controlled, the gain from σ to z_j cannot be tuned by feedback control. In other words, due to this system structure, only the gain from σ to the x_j-subsystem can be tuned. Therefore, it becomes impossible to achieve a (sufficiently small) input-to-output gain for the complete closed-loop system with mixup of z_j and x_j-subsystems unless they are separately considered at each recursive step. It is technically challenging to separately consider the two types of subsystems due to the complicated nonlinear coupling between them. Therefore, this paper contributes to develop a new recursive design tool that is able to accommodate this scenario. Obviously, the aforementioned technical challenge appears only in applying recursive input-to-output stabilization and gain calculation for systems of high relative degrees. It makes the contribution of this paper significant over the study on the $r = 1$ case in Zhu et al. (2016) that does not rely on recursive design.

It is worth mentioning that the input-to-output stabilization problem is closely related to H_∞ control of nonlinear systems, where σ is regarded as the disturbance input (Ball, Helton, & Walker, 1993; Isidori & Kang, 1995; Van der Schaft, 1992). The solution of nonlinear H_∞ control relies on the solution of a nonlinear Hamilton–Jacobi–Isaacs partial differential equation (HJIE). Although the formulation of nonlinear H_∞ control has been well developed, the analytical solution of HJIE remains challenging for practical applications. In this paper, we are interested in finding an alternative explicit design approach for the input-to-output stabilization problem.

2. Problem formulation

The *robust input-to-output stabilization problem* aims to design a controller for the lower triangular system (1) such that the trajectories of the closed-loop system satisfy the robust input-to-state stable (RISS) property[1]

$$|\xi(t)| \le \max\{\beta_o(|\xi(t_o)|, t - t_o), \gamma_o(\|\sigma_{[t_o,t]}\|)\}, \tag{2}$$

and the robust input-to-output stable (RIOS) property

$$|y(t)| \le \max\{\beta(|\xi(t_o)|, t - t_o), \gamma(\|\sigma_{[t_o,t]}\|)\} \tag{3}$$

for some $\mathcal{KL}$ functions β_o and β and class $\mathcal{K}$ functions γ_o and γ. Moreover, the RIOS gain γ satisfies

$$\gamma(s) < \gamma_c(s), \ \ s > 0 \tag{4}$$

for a specified class $\mathcal{K}$ function γ_c. Obviously, when the external perturbation σ vanishes, the aforementioned robust input-to-output stabilization problem reduces to the traditional robust stabilization problem. The inclusion of a specified gain inequality (4) brings a major challenge as discussed in Introduction, in particular, when the system contains dynamic uncertainties.

The technical challenge for the general problem with $r > 1$ is the development of a new recursive design. For this purpose, we first introduce the following coordinate transformation

$$\begin{aligned} \bar{x}_1 &= x_1 \\ \bar{x}_i &= x_i - \nu_{i-1}(\bar{x}_{i-1}), \ \ i = 2, \ldots, r+1 \end{aligned} \tag{5}$$

where functions $\nu_1, \ldots, \nu_r$ satisfying $\nu_i(0) = 0$ represent virtual controllers to be designed at each recursive step. Obviously, letting $\bar{x}_{r+1} = 0$ gives the controller $u = x_{r+1} = \nu_r(\bar{x}_r)$. Under the coordinate (5), the system (1) becomes

$$\begin{aligned} \dot{z}_j &= \bar{q}_j(\vec{z}_j, \vec{\bar{x}}_j, \sigma, d) \\ \dot{\bar{x}}_j &= \bar{f}_j(\vec{z}_j, \vec{\bar{x}}_j, \sigma, d) + b_j(d)(\bar{x}_{j+1} + \nu_j(\bar{x}_j)), \ \ j = 1, \ldots, r \end{aligned} \tag{6}$$

where $\vec{\bar{x}}_j := \mathrm{col}(\bar{x}_1, \ldots, \bar{x}_j)$, and the functions $\bar{q}_j(\vec{z}_j, \vec{\bar{x}}_j, \sigma, d)$ and $\bar{f}_j(\vec{z}_j, \vec{\bar{x}}_j, \sigma, d)$ are sufficiently smooth satisfying $\bar{q}_j(0, 0, 0, d) = 0$ and $\bar{f}_j(0, 0, 0, d) = 0$. Denote the full state of (6) as $\bar{\xi} := \mathrm{col}(\vec{z}_r, \vec{\bar{x}}_r)$.

Now, the solvability of the aforementioned robust input-to-output stabilization problem can be investigated based on the following proposition.

Proposition 2.1. *Consider the system (1) with the coordinate transformation (5). For a given class $\mathcal{K}$ function γ_c, suppose there exist functions $\nu_1, \ldots, \nu_r$ such that the trajectories of the closed-loop system (6) with $\bar{x}_{r+1} = 0$ satisfy*

$$|\vec{z}_r(t)| \le \max\{\beta_z(|\bar{\xi}(t_o)|, t - t_o), \gamma_z(\|\sigma_{[t_o,t]}\|)\} \tag{7}$$

$$|\vec{\bar{x}}_r(t)| \le \max\{\beta_x(|\bar{\xi}(t_o)|, t - t_o), \gamma_x(\|\sigma_{[t_o,t]}\|)\} \tag{8}$$

for $\beta_z, \beta_x \in \mathcal{KL}$ and $\gamma_z, \gamma_x \in \mathcal{K}$ with

$$\gamma_x(s) < \gamma_c(s), \ \ s > 0. \tag{9}$$

Then, the robust input-to-output stabilization problem with the specified gain γ_c is solved in the sense of (2)–(4). ■

Proof. By the fact $|\bar{\xi}| = |\vec{z}_r| + |\vec{\bar{x}}_r|$, (7) and (8) imply

$$|\bar{\xi}(t)| \le \max\{\bar{\beta}_o(|\bar{\xi}(t_o)|, t - t_o), \bar{\gamma}_o(\|\sigma_{[t_o,t]}\|)\} \tag{10}$$

for $\bar{\beta}_o \in \mathcal{KL}$ and $\bar{\gamma}_o \in \mathcal{K}$, which is equivalent to (2) according to the mapping (5) between ξ and $\bar{\xi}$. Also, simple calculation shows $|y| = |\bar{x}_1| \le |\vec{\bar{x}}_r|$, which, together with (8), implies (3) with $\gamma = \gamma_x$. Finally, (4) is straightforward from (9). ■

[1] In the definitions of RISS and RIOS, it is implicitly assumed that the input function $\sigma : [t_0, \infty) \mapsto \mathbb{R}^{n_\sigma}$ is piecewise continuous and bounded with the supremum norm $\|\sigma_{[t_0,\infty)}\| = \sup_{t \ge t_0} |\sigma(t)|$. For convenience, we also denote the supremum norm of the truncation of $\sigma(t)$ in $[t_1, t_2]$ as $\|\sigma_{[t_1,t_2]}\| = \sup_{t_1 \le t \le t_2} |\sigma(t)|$. The notation $|\sigma|$ means the Euclidean norm of σ.