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Equivalent types of ISS Lyapunov functions for discontinuous discrete-time systems[☆]

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ABSTRACT

In this note, we study *input-to-state stability* (ISS) of discontinuous discrete-time systems via ISS Lyapunov functions (ISS LFs). For *continuous* discrete-time systems it is well-known that the existence of a dissipation-form ISS LF is equivalent to the existence of an implication-form ISS LF. For *discontinuous* discrete-time systems it was recently shown that this equivalence is no longer satisfied, and a stronger definition of an implication-form ISS LF is introduced. Moreover, we consider max-form ISS LFs. Here, we give a sufficient and necessary condition under which the existence of all these three forms of ISS LFs are equivalent in the sense that the existence of each of these forms implies the existence of both the other forms. Most importantly, this condition, called *global $\mathcal{K}$-boundedness*, is shown to be also a necessary condition for ISS. To give a complete characterization we consider the case of ISS with respect to two measurement functions, which includes ISS with respect to a single measure and classic ISS as special cases.

1. Introduction

In the stability analysis of nonlinear control systems, the concept of *input-to-state stability* as introduced in Sontag (1989) has been shown to serve as an efficient tool. One reason for the enormous popularity of ISS in nonlinear control is its characterization via the existence of ISS LFs. Whereas the first papers on ISS dealt with *continuous-time* systems, the definition and characterizations via ISS LFs for *discrete-time* systems, as considered in this work, also have been derived, see Grüne and Kellett (2014), Jiang and Wang (2001), Kazakos and Tsinias (1994) and Lazar, Heemels, and Teel (2009).

In general, there are two types of ISS LFs that are mainly used; an implication-form and a dissipation-form ISS LF. For *continuous* discrete-time systems it was shown in Jiang and Wang (2001) that these two forms are equivalent in the sense that the existence of an implication-form ISS LF implies the existence of a dissipation-form ISS LF and vice versa. However, the authors in Grüne and Kellett (2014) showed that for *discontinuous* discrete-time systems such as e.g. dynamical systems controlled via optimization-based techniques, there is no equivalence between these two forms. In particular, the existence of an implication-form ISS LF is not sufficient to prove ISS of a discontinuous discrete-time system. To overcome this problem, the authors introduced the concept of a *strong* implication-form ISS LF, which requires an additional condition of the implication-form ISS LF. Although this additional condition does not appear to be hard to check, the question remains open under which conditions on the *system dynamics* the existence of an implication-form ISS LF yields ISS.

This question will be answered in this work. The key ingredient is a property of the dynamics called *global $\mathcal{K}$-boundedness*, see Geiselhart, Gielen, Lazar, and Wirth (2014). As shown in Geiselhart and Wirth (2016), the global $\mathcal{K}$-boundedness property is even a *necessary* condition for ISS of a discrete-time system. Moreover, we are able to prove that global $\mathcal{K}$-boundedness is sufficient *and* necessary for the equivalence between implication-form and dissipation-form ISS LFs to hold. In addition, we show that under the global $\mathcal{K}$-boundedness, any implication-form ISS LF is also strong implication-form ISS LF. This observation is rather useful for the application of the small-gain result in Grüne and Sigurani (2014), where strong implication-form ISS LF estimates yield *less* conservative gains. In addition, we also prove equivalence of any of these ISS LFs to a so-called max-form ISS LF. This form of an ISS LF is particularly relevant for large-scale discrete-time systems as shown in Noroozi, Geiselhart, Grüne, Rüffer, and Wirth (2017). As far as we know, the question concerning the comparison of

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conservativeness of gains has not been investigated so far. We will not treat this problem here, but note that the computation of *tight* gains was studied in Zhang and Dower (2013). Moreover, a comparison would require some *measure of conservativeness*, for instance, the guaranteed decay rate of the complete system, Borgers, Geiselhart, and Heemels (2017).

Instead of considering stability properties with respect to a norm, the idea of considering stability with respect to measurement functions has a long history, see e.g. Lakshmikantham and Liu (1993), Movchan (1960). Extensions toward the concept of ISS, i.e., ISS with respect to measurement functions, include, for instance, the concept of Input-to-Output Stability (Jiang, Lin, & Wang, 2005). For discrete-time systems, Lyapunov function characterizations of ISS with respect to one (*single-measure* case) or two (*two-measure* case) measurement functions have been proposed in Tran, Kellett, and Dower (2015a, b). In this note, we derive our results for the two-measure case. Conclusions for the single-measure and the classic ISS case can be drawn easily. This extends previous results in several ways. First, we drop the commonly used continuity assumption on the dynamics. Second, under a global $\mathcal{K}$-boundedness assumption, we prove the equivalence between the above mentioned different forms of (ω_1, ω_2)ISS LFs and ISS of the system with respect to the measurement functions ω_1, ω_2. Third, we show that the assumption that the measurement function is a proper indicator function is *not* required, provided the global $\mathcal{K}$-boundedness property, which is necessary for (ω_1, ω_2)ISS, holds.

The outline of the paper is as follows. We start with some preliminary notation and the problem statement in Sections 2 and 3, respectively. The main results of this note are given in Section 4. Conclusions are drawn in Section 5.

2. Preliminaries

By $\mathbb{N}$ we denote the natural numbers including 0. Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{R}^n$ the vector space of real column vectors of length n; further $\mathbb{R}^n_+$ denotes the positive orthant. A partial order for vectors $v, w \in \mathbb{R}^n$ is defined by $v \geq w$ (resp. $v > w$) if and only if $[v]_i \geq [w]_i$ (resp. $[v]_i > [w]_i$), where $[v]_i$ is the ith component of v, $i \in \{1, \ldots, n\}$. By $\|\cdot\|$ we denote an arbitrary monotonic norm on $\mathbb{R}^n$, i.e. if $v, w \in \mathbb{R}^n_+$ with $w \geq v$, then $\|w\| \geq \|v\|$. For a sequence $\{u(k)\}_{k\in\mathbb{N}}$ with $u(k) \in \mathbb{R}^m$, we define $\|u\|_\infty := \sup_{k\in\mathbb{N}}\{\|u(k)\|\} \in \mathbb{R}_+ \cup \{\infty\}$. If $u(\cdot)$ is bounded, i.e., $\|u\|_\infty < \infty$, then $u(\cdot) \in \ell^\infty(\mathbb{R}^m)$.

By the equivalence of norms there exists, for any norm $\|\cdot\|$ on $\mathbb{R}^n$, a constant $\kappa \geq 1$ satisfying for all $x = (x_1, \ldots, x_N) \in \mathbb{R}^n$ with $x_i \in \mathbb{R}^{n_i}$ and $n = \sum_{i=1}^N n_i$,

$$\|x\| \leq \kappa \max_{i\in\{1,\ldots,N\}} \|x_i\| \,, \tag{1}$$

where $\|x_i\| := \|0, \ldots, 0, x_i, 0, \ldots, 0\|$. Note that for any p-norm $\|\cdot\|$ the smallest κ satisfying (1) is $\kappa = N^{1/p}$.

Throughout this work, standard *comparison functions* $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ are used: The class $\mathcal{K}$ consists of all continuous functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, which are strictly increasing and zero in zero. The class $\mathcal{K}_\infty$ is a subclass of class $\mathcal{K}$, which consists of all $\mathcal{K}$-function that are unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$, if it is of class $\mathcal{K}$ in the first argument and decreasing to zero in the second argument. Moreover, we call a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *positive (semi)definite*, if $\alpha(0) = 0$ and $\alpha(s) > 0$ (resp. $\alpha(s) \geq 0$) for all $s > 0$. We denote the class of positive definite functions by $\mathcal{P}$.

3. Problem statement and preliminary results

We consider discrete-time systems of the form

$$x(k+1) = G(x(k), u(k)), \qquad k \in \mathbb{N}, \tag{2}$$

with state $x(k) \in \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ satisfying $G(0, 0) = 0$. Here $u(k) \in \mathbb{R}^m$ denotes the input at time $k \in \mathbb{N}$. Note that an input is a function $u : \mathbb{N} \to \mathbb{R}^m$. By $x(k, \xi, u(\cdot)) \in \mathbb{R}^n$ we denote the solution of the discrete-time system (2) at time $k \in \mathbb{N}$ with initial state $x(0) = \xi \in \mathbb{R}^n$ and input $u(\cdot)$.

Let $\omega : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous and positive semidefinite function, which we refer to as a *measurement function*. Below, we derive our results for the *two-measure* case, i.e., we define input-to-state stability with respect to two measurement functions ω_1 and ω_2. If both measurement functions ω_1 and ω_2 are equal, we talk about the *single-measure* case and we set $\omega := \omega_1 = \omega_2$. In particular, if $\omega(\cdot) = \|\cdot\|$ we obtain the classic ISS case, and we omit the reference to the measurement function.

Definition 1. We call system (2) *input-to-state stable* with respect to the measurement functions ω_1, ω_2 ((ω_1, ω_2)ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $\xi \in \mathbb{R}^n$, any input $u(\cdot) \in \ell^\infty(\mathbb{R}^m)$, and any $k \in \mathbb{N}$,

$$\omega_1(x(k, \xi, u(\cdot))) \leq \beta(\omega_2(\xi), k) + \gamma(\|u\|_\infty). \tag{3}$$

In the single-measure case, we call system (2) ωISS, if (3) is satisfied for $\omega_1 = \omega_2 = \omega$.

Remark 2. ISS notions with respect to one or two measurement functions include the classic ISS case ($\omega_1(\cdot) = \omega_2(\cdot) = \|\cdot\|$), the case of Input-to-Output Stability (IOS) ($\omega_1(\cdot) = \|h(\cdot)\|$ with output map h and $\omega_2(\cdot) = \|\cdot\|$) or State-Independent Input-to-Output Stability ($\omega_1(\cdot) = \omega_2(\cdot) = \|h(\cdot)\|$).

We emphasize that we do not require any regularity condition on G. However, we define the concept of global $\mathcal{K}$-boundedness, which is crucial for this work.

Definition 3. The function $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ in (2) is globally $\mathcal{K}$-bounded with respect to (ω_1, ω_2), if there exist functions $\eta_1, \eta_2 \in \mathcal{K}$ such that for all $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$, we have

$$\omega_1(G(\xi, \mu)) \leq \eta_1(\omega_2(\xi)) + \eta_2(\|\mu\|). \tag{4}$$

We call G globally $\mathcal{K}$-bounded with respect to ω, if $\omega_1 = \omega_2 =: \omega$. When the measurement functions are clear from the context, we simply say that G is *globally $\mathcal{K}$-bounded*.

Note that, in order to prove ISS, the global $\mathcal{K}$-boundedness property is indeed necessary, as shown in Geiselhart and Wirth (2016, Remark 3.3) for the case $\omega(\cdot) = \|\cdot\|$. But this fact can be easily shown for the general case of ISS with respect to one or two measures. It is important to note that Lemma 4 shows that global $\mathcal{K}$-boundedness is no restriction when studying (ω_1, ω_2)ISS of system (2).

Lemma 4. *If system (2) is (ω_1, ω_2)ISS then G in (2) is globally $\mathcal{K}$-bounded with respect to (ω_1, ω_2).*

Proof. By definition, any $\mathcal{KL}$-function β is of class $\mathcal{K}$ in the first argument. Moreover, as $G(\xi, \mu) = x(1, \xi, \mu)$, the ISS estimate (3) directly yields

$$\omega_1(G(\xi, \mu)) = \omega_1(x(1, \xi, \mu)) \leq \beta(\omega_2(\xi), 1) + \gamma(\|\mu\|),$$

which gives the global $\mathcal{K}$-boundedness property (4) with $\eta_1(\cdot) = \beta(\cdot, 1)$ and $\eta_2(\cdot) = \gamma(\cdot)$. □

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