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A theory of passive linear systems with no assumptions[☆]

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ABSTRACT

We present two linked theorems on passivity: the passive behavior theorem, parts 1 and 2. Part 1 provides necessary and sufficient conditions for a general linear system, described by a set of high order differential equations, to be passive. Part 2 extends the positive-real lemma to include uncontrollable and unobservable state-space systems.

1. Introduction

A system is called *passive* if there is an upper bound on the net energy that can be extracted from the system from the present time onwards. This is a fundamental property of many physical systems. In systems and control theory, the concept of passivity has its origins in the study of electric networks comprising resistors, inductors, capacitors, transformers, and gyrators (RLCTG networks). In contemporary systems theory, passive systems are more familiar through their role in the positive-real lemma. This lemma proves the equivalence of: (i) an integral condition related to the energy exchanged with the system; (ii) a condition on the transfer function for the system (the positive-real condition); and (iii) a linear matrix inequality involving the matrices in a state-space realization for the system. As well as being relevant to passive systems, the lemma also gives necessary and sufficient conditions for the existence of non-negative definite solutions to an important linear matrix inequality and algebraic Riccati equation, and has links with spectral factorization. However, these results are all subject to one caveat: the system is assumed to be controllable.

As emphasized by Çamlibel, Willems, and Belur (2003), Hughes and Smith (2017) and Willems (2007), there is no explicit connection between the concepts of passivity and controllability. Moreover, the a-priori assumption of controllability in the positive-real lemma leaves open several questions of physical significance. In particular, it is not known what uncontrollable behaviors can be realized as the driving-point behavior of an electric (RLCTG) network. Similarly, necessary and sufficient conditions for the existence of a non-negative definite solution to the linear matrix inequality (and algebraic Riccati equation) considered in the positive-real lemma are unknown when the state-space realization under consideration is uncontrollable. There have been many papers in the literature that have aimed to relax the assumption of controllability in the positive-real lemma, e.g., Collado, Lozano, and Johansson (2001), Kunimatsu, Sang-Hoon, Fujii, and Ishitobi (2008) and Pandolfi (2001) (and many papers have studied uncontrollable cyclo-dissipative systems, e.g., Çamlibel et al., 2003; Ferrante and Pandolfi, 2002; Ferrante, 2005; Pal and Belur, 2008), but all of these papers contain other a-priori assumptions. The objective of this paper is to provide a complete theory of passive linear systems with no superfluous assumptions. Our main contributions are: 1. a new trajectory-based definition of passivity (Definition 5); and 2. two linked theorems that we call the *passive behavior theorem*, parts 1 and 2. Part 1 (Theorem 9) provides necessary and sufficient conditions for the passivity of a general linear system (described by a differential equation of the form $P(\frac{d}{dt})\mathbf{i} = Q(\frac{d}{dt})\mathbf{v}$ for some square polynomial matrices P and Q). This generalizes classical results that are restricted to controllable behaviors (where P and Q are left coprime). Part 2 (Theorem 13) extends the positive-real lemma by removing the a-priori controllability and observability assumptions. As a corollary of these results, we find that any passive (not necessarily controllable) behavior can be realized as the driving-point behavior of an electric (RLCTG) network.

The structure of the paper is as follows. In Section 2, we discuss the positive-real lemma and its limitations. Section 3 discusses our

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new definition of *passivity*. Then, in Section 4, we introduce the new concept of a *positive-real pair*, and we state our two passive behavior theorems. It is shown that our new concept of a positive-real pair provides the appropriate extension of the positive-real concept to uncontrollable systems. Specifically, for any pair of square polynomial matrices P and Q, we show that the system corresponding to the solutions to the differential equation $P(\frac{d}{dt})\mathbf{i} = Q(\frac{d}{dt})\mathbf{v}$ is passive if and only if (P, Q) is a positive-real pair. The proofs of the passive behavior theorems are in Section 6, and some preliminary results appear in Section 5. Finally, the paper is strongly influenced by the behavioral approach to dynamical systems (see Polderman & Willems, 1998). Therefore, to make the paper accessible to the reader unfamiliar with behavioral theory, we provide four short appendices containing relevant background on linear systems, behaviors, and polynomial matrices. These contain numbered notes (A1, A2, and so forth) that will be referred to in the text. The reader who wishes to follow the proofs in Sections 5 and 6 is advised to first read these appendices.

The notation is as follows. $\mathbb{R}$ ($\mathbb{C}$) denotes the real (complex) numbers; $\mathbb{C}_+$ ($\bar{\mathbb{C}}_+$) denotes the open (closed) right-half plane; $\mathbb{C}_-$ ($\bar{\mathbb{C}}_-$) denotes the open (closed) left-half plane. $\mathbb{R}[\xi]$ ($\mathbb{R}(\xi)$) denotes the polynomials (rational functions) in the indeterminate ξ with real coefficients. $\mathbb{R}^{m\times n}$ (resp., $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}[\xi]$, $\mathbb{R}^{m\times n}(\xi)$) denotes the matrices with m rows and n columns with entries from $\mathbb{R}$ (resp., $\mathbb{C}$, $\mathbb{R}[\xi]$, $\mathbb{R}(\xi)$), and the number n is omitted whenever $n = 1$. If $H \in \mathbb{C}^{m\times n}$, then $\Re(H)$ ($\Im(H)$) denotes its real (imaginary) part, and $\bar{H}$ its complex conjugate. If $H \in \mathbb{R}^{m\times n}, \mathbb{C}^{m\times n}, \mathbb{R}^{m\times n}[\xi]$ or $\mathbb{R}^{m\times n}(\xi)$, then H^T denotes its transpose; and if H is nonsingular (i.e., $\det(H) \not\equiv 0$), then H^{-1} denotes its inverse. We let $\mathrm{col}(H_1 \ \cdots \ H_n)$ ($\mathrm{diag}(H_1 \ \cdots \ H_n)$) denote the block column (block diagonal) matrix with entries $H_1, \ldots, H_n$. If $M \in \mathbb{C}^{m\times m}$, then $M > 0$ ($M \geq 0$) indicates that M is Hermitian positive (non-negative) definite, and $\mathrm{spec}(M) := \{\lambda \in \mathbb{C} \mid \det(\lambda I - M) = 0\}$. If $G \in \mathbb{R}^{m\times n}(\xi)$, then $\mathrm{normalrank}(G) := \max_{\lambda\in\mathbb{C}}(\mathrm{rank}(G(\lambda)))$, $G^\star(\xi) := G(-\xi)^T$, G is called *para-Hermitian* if $G = G^\star$, and *proper* if $\lim_{\xi\to\infty}(G(\xi))$ exists. $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ and $\mathcal{C}_\infty(\mathbb{R}, \mathbb{R}^k)$ denote the (k-vector-valued) locally integrable and infinitely-often differentiable functions (Polderman & Willems, 1998 Definitions 2.3.3, 2.3.4). We equate any two locally integrable functions that differ only on a set of measure zero. If $\mathbf{w} \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$, then $\mathbf{w}^T$ denotes the function satisfying $\mathbf{w}^T(t) = \mathbf{w}(t)^T$ for all $t \in \mathbb{R}$. We also consider the function space

$$\mathcal{E}_{\mathbb{C}_-}(\mathbb{R}, \mathbb{R}^k) := \left\{ \mathbf{w} \mid \mathbf{w}(t) = \Re\left(\sum_{i=1}^{N}\sum_{j=0}^{n_i-1} \tilde{\mathbf{w}}_{ij} t^j e^{\lambda_i t}\right) \text{ for all } t \in \mathbb{R} \right.$$
$$\left. \text{with } \tilde{\mathbf{w}}_{ij} \in \mathbb{C}^k, \lambda_i \in \mathbb{C}_-, \text{ and } N, n_i \text{ integers} \right\},$$

and note that $\mathcal{E}_{\mathbb{C}_-}(\mathbb{R}, \mathbb{R}^k) \subset \mathcal{C}_\infty(\mathbb{R}, \mathbb{R}^k) \subset \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$.

We consider behaviors (systems) defined as the set of *weak* solutions to a linear differential equation:

$$\mathcal{B} = \{\mathbf{w} \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k) \mid R(\tfrac{d}{dt})\mathbf{w} = 0\}, \ R \in \mathbb{R}^{l\times k}[\xi]. \tag{1.1}$$

Here, if $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L$ and $\mathbf{w} \in \mathcal{C}_\infty(\mathbb{R}, \mathbb{R}^k)$, then $R(\frac{d}{dt})\mathbf{w} = R_0\mathbf{w} + R_1\frac{d\mathbf{w}}{dt} + \cdots + R_L\frac{d^L\mathbf{w}}{dt^L}$ (see Polderman & Willems, 1998 Definition 2.3.7 for the meaning of a weak solution to $R(\frac{d}{dt})\mathbf{w} = 0$ when $\mathbf{w}$ is not necessarily differentiable). Particular attention is paid to the special class of state-space systems:

$$\mathcal{B}_s = \{(\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d)$$
$$\text{such that } \tfrac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u} \text{ and } \mathbf{y} = C\mathbf{x} + D\mathbf{u}\},$$
$$\text{with } A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times n}, C \in \mathbb{R}^{n\times d}, D \in \mathbb{R}^{n\times n}. \tag{1.2}$$

Several properties of state-space systems are listed in Appendix D. In particular, from note D1, if $(\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{B}_s$, then $\mathbf{x}$ satisfies the variation of the constants formula almost everywhere, which determines the value $\mathbf{x}(t_1)$ of $\mathbf{x}$ at an instant $t_1 \in \mathbb{R}$. Finally, we also consider behaviors obtained by permuting and/or eliminating variables in a behavior $\mathcal{B}$ as in (1.1). For example, associated with the state-space system $\mathcal{B}_s$ in (1.2) is the corresponding external behavior $\mathcal{B}_s^{(\mathbf{u},\mathbf{y})} = \{(\mathbf{u}, \mathbf{y}) \mid \exists \mathbf{x} \text{ with } (\mathbf{u}, \mathbf{y}, \mathbf{x}) \in \mathcal{B}_s\}$. More generally, for any given $T_1 \in \mathbb{R}^{l_1\times k}, \ldots, T_n \in \mathbb{R}^{l_n\times k}$ such that $\mathrm{col}(T_1 \ \cdots \ T_n) \in \mathbb{R}^{k\times k}$ is a permutation matrix, and integer $1 \leq m \leq n$, we denote the projection of $\mathcal{B}$ onto $T_1\mathbf{w}, \ldots, T_m\mathbf{w}$ by

$$\mathcal{B}^{(T_1\mathbf{w},\ldots,T_m\mathbf{w})} = \{(T_1\mathbf{w}, \ldots, T_m\mathbf{w}) \mid \exists (T_{m+1}\mathbf{w}, \ldots, T_n\mathbf{w})$$
$$\text{such that } \mathbf{w} \in \mathcal{B}\}.$$

2. The positive-real lemma

The central role of passivity in systems and control is exemplified by the positive-real lemma (see Lemma 1). The name positive-real (PR) describes a function $G \in \mathbb{R}^{n\times n}(\xi)$ with the properties: (i) G is analytic in $\mathbb{C}_+$; and (ii) $G(\bar{\lambda})^T + G(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$ (see Anderson & Vongpanitlerd, 1973 Theorem 2.7.2 for a well known equivalent condition). The positive-real lemma then considers a state-space system as in (1.2) and provides necessary and sufficient conditions for the transfer function $G(\xi) = D + C(\xi I - A)^{-1}B$ to be PR. Notably, it is assumed that (A, B) is controllable and (C, A) is observable (see notes D2 and D4).

Lemma 1 (*Positive-Real Lemma*)**.** *Let $\mathcal{B}_s$ be as in (1.2) and let (A, B) be controllable and (C, A) observable. Then the following are equivalent:*

1. *Given any $\mathbf{x}_0 \in \mathbb{R}^d$, there exists $S_a(\mathbf{x}_0) \in \mathbb{R}$ with* $S_a(\mathbf{x}_0) := \sup_{\substack{t_1 \geq t_0 \in \mathbb{R}, (\mathbf{u},\mathbf{y},\mathbf{x}) \in \mathcal{B}_s \\ \text{with } \mathbf{x}(t_0) = \mathbf{x}_0}} \left(-\int_{t_0}^{t_1} \mathbf{u}^T(t)\mathbf{y}(t)dt\right)$.
2. $\sup_{\substack{t_1 \geq t_0 \in \mathbb{R}, (\mathbf{u},\mathbf{y},\mathbf{x}) \in \mathcal{B}_s \\ \text{with } \mathbf{x}(t_0) = 0}} \left(-\int_{t_0}^{t_1} \mathbf{u}^T(t)\mathbf{y}(t)dt\right) = 0$
3. *There exist real matrices X, L_X, W_X such that $X > 0$, $-A^TX - XA = L_X^TL_X$, $C - B^TX = W_X^TL_X$, and $D + D^T = W_X^TW_X$.*
4. *$G(\xi) := D + C(\xi I - A)^{-1}B$ is PR.*

If, in addition, $D+D^T > 0$, then the above conditions are equivalent to:

5. *There exists a real $X > 0$ such that $\Pi(X) := -A^TX - XA - (C^T - XB)(D + D^T)^{-1}(C - B^TX) = 0$ and $\mathrm{spec}(A + B(D + D^T)^{-1}(B^TX - C)) \in \bar{\mathbb{C}}_-$.*

For a proof of the positive-real lemma, we refer to Anderson and Vongpanitlerd (1973) and Willems (1972b). These references also describe links with spectral factorization, which is the concern of the following well known result (Youla, 1961 Theorem 2):

Lemma 2 (*Youla's Spectral Factorization Result*)**.** *Let $H \in \mathbb{R}^{n\times n}(\xi)$ be para-Hermitian; let $H(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$, ω not a pole of H; and let $\mathrm{normalrank}(H) = r$. There exists a $Z \in \mathbb{R}^{r\times n}(\xi)$ such that* (i) *$H = Z^\star Z$;* (ii) *Z is analytic in $\mathbb{C}_+$; and* (iii) *$Z(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}_+$. Moreover, if $H \in \mathbb{R}^{n\times n}[\xi]$, then $Z \in \mathbb{R}^{r\times n}[\xi]$; if $H(j\omega)$ is analytic for all $\omega \in \mathbb{R}$, then Z is analytic in $\bar{\mathbb{C}}_+$; and if $Z_1 \in \mathbb{R}^{r\times n}(\xi)$ also satisfies* (i)–(iii)*, then there exists a $T \in \mathbb{R}^{r\times r}$ such that $Z_1 = TZ$ and $T^TT = I$. We call any $Z \in \mathbb{R}^{r\times n}(\xi)$ that satisfies* (i)–(iii) *a* spectral factor *of H.*

Remark 3. When G is as in Lemma 1 with $D + D^T > 0$, there exists $W_X \in \mathbb{R}^{n\times n}$ with $D + D^T = W_X^TW_X$. Then, with X as in condition 5 of Lemma 1, it can be shown that $Z_X(\xi) := W_X + (W_X^T)^{-1}(C - B^TX)(\xi I - A)^{-1}B$ is a spectral factor of $G + G^\star$ (see Willems, 1972b).