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Sparse Jurdjevic–Quinn stabilization of dissipative systems ☆

Marco Caponigro [a], Benedetto Piccoli [b], Francesco Rossi [c], Emmanuel Trélat [d]

[a] Conservatoire National des Arts et Métiers, Équipe M2N, 292 rue Saint-Martin, 75003, Paris, France
[b] Department of Mathematical Sciences and Center for Computational and Integrative Biology, Rutgers University, Camden, NJ 08102, USA
[c] Aix Marseille Université, CNRS, ENSAM, Université de Toulon, LSIS, Marseille, France
[d] Sorbonne Universités, UPMC Univ. Paris 6, CNRS UMR 7598, Laboratoire Jacques-Louis Lions and Institut Universitaire de France, 4 place Jussieu, 75005, Paris, France

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ABSTRACT

For control-affine systems with a proper Lyapunov function, the classical Jurdjevic–Quinn procedure (see Jurdjevic and Quinn, 1978) gives a well-known and widely used method for the design of feedback controls that asymptotically stabilize the system to some invariant set. In this procedure, all controls are in general required to be activated, i.e. nonzero, at the same time.

In this paper we give sufficient conditions under which this stabilization can be achieved by means of sparse feedback controls, i.e., feedback controls having the smallest possible number of nonzero components. We thus obtain a sparse version of the classical Jurdjevic–Quinn theorem.

We propose three different explicit stabilizing control strategies, depending on the method used to handle possible discontinuities arising from the definition of the feedback: a time-varying periodic feedback, a sampled feedback, and a hybrid hysteresis. We illustrate our results by applying them to opinion formation models, thus recovering and generalizing former results for such models.

1. Introduction and main result

1.1. The context

Let n and m be positive integers, let f and g_i, $i = 1, \ldots, m$ be smooth vector fields defined on $\mathbb{R}^n$, and let $\mathbb{U}$ be a convex subset of $\mathbb{R}^m$ containing a neighborhood of the origin. We consider the control-affine system in $\mathbb{R}^n$

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)), \tag{1}$$

where the control $u = (u_1, \ldots, u_m)$ takes its values in $\mathbb{U}$. We assume the uncontrolled system (i.e., with $u \equiv 0$) to be *dissipative*, meaning that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- V is radially unbounded (or proper), i.e. $V^{-1}((-\infty, \ell])$ is compact for every $\ell \in \mathbb{R}$;
- $L_f V(x) \leq 0$ for every $x \in \mathbb{R}^n$.

According to the well-known Jurdjevic–Quinn theorem (see Jurdjevic & Quinn, 1978), if we assume that $f(0) = 0$ and that

$$\{0\} = \{x \in \mathbb{R}^n \mid L_f V(x) = 0 \text{ and } L_f^k L_{g_i} V(x) = 0, \\ \text{for } i = 1, \ldots, m,\ k \in \mathbb{N}\},$$

then the smooth feedback defined by

$$u(x) = -(L_{g_1} V(x), L_{g_2} V(x), \ldots, L_{g_m} V(x)) \tag{2}$$

globally asymptotically stabilizes the system (1) to 0. A more general version gives the convergence to some invariant set. The convergence is established by the LaSalle invariance principle. This famous result has been widely used, in various contexts, ranging from the control of mechanical systems (see for instance Faubourg & Pomet, 1999, 2000; Outbib & Sallet, 1992) to mathematical biology (see, e.g., Auger et al., 2009).

In the above strategy, all components of the control are in general active i.e., they take non-zero values. We address here the following question: is it possible to design a similar Jurdjevic–Quinn stabilizing feedback strategy in which only a minimal number of controls are active at each instant of time?

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E-mail addresses: marco.caponigro@cnam.fr (M. Caponigro), piccoli@camden.rutgers.edu (B. Piccoli), francesco.rossi@lsis.org (F. Rossi), emmanuel.trelat@upmc.fr (E. Trélat).

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This question is inspired by the works (Caponigro, Fornasier, Piccoli, & Trélat, 2013, 2015) introducing the notion of *sparse control*. The term "sparse" may refer to components or to time.

A componentwise-sparse control has only at most one active component at each instant of time. Componentwise sparsity is motivated by many applications: when dealing with high-dimensional problems, that is when both $n \gg 1$ and $m \gg 1$, it may be inadequate to implement a control having m active components. It is therefore natural to seek controls achieving the same goal with less active components. This is the case for instance when we want only one leader to act on a whole crowd (such as a dog with a flock of sheep), or more generally when feasible control strategies are required to focus on a small number of agents at each time (see Albi, Bongini, Cristiani, & Kalise, 2016; Albi & Pareschi, 2013; Bongini, Fornasier, Rossi, & Solombrino, accepted; Borzì & Wongkaew, 2015; Fornasier, Piccoli, & Rossi, 2014; Wongkaew, Caponigro, & Borzì, 2015).

A problem for such a componentwise-sparse control is that it may chatter, i.e., it may change active component infinitely many times over a compact time-interval; such a chattering phenomenon may cause some theoretical, numerical and practical difficulties. In particular, chattering is an obstacle to well-posedness and convergence of numerical schemes (see Zhu, Trélat, & Cerf, 2016). Time-sparsity was then introduced in Caponigro et al. (2013, 2015) to avoid these unwanted phenomena. A time-sparse control, indeed, has a minimal gap between two switchings. In this paper, we enforce time-sparsity by using either time-sampling or hysteresis.

The motivation that we have in mind is to address the control of large groups of interacting agents, by means of control strategies that are both as simple and sparse as possible. In Section 3.2, we will then test the sparse control strategies that we develop throughout on classical examples of opinion dynamics.

1.2. Sparse feedback stabilization strategies

We provide hereafter three different control strategies to achieve stabilization by using a sparse Jurdjevic–Quinn controller, mimicking the form (2). Starting from this idea, our aim is to achieve sparse stabilization, by choosing sparse controls of the form $u_i(x) = -L_{g_i}V(x)$ for some $i \in \{1, \ldots, m\}$, while $u_j(x) = 0$ for $j \neq i$. The key aspect for achieving sparse stabilization is to determine the strategy to switch from one active component of the control to another one. Indeed, discontinuity issues in the definition of sparse stabilizers arise naturally, as shown for instance in Bongini, Fornasier, Fröhlich, Haghverdi, et al. (2014), Caponigro et al. (2013, 2015) and Caponigro, Piccoli, Rossi, and Trélat (2016), see also Section 3.1. Here we develop three different approaches to deal with discontinuous feedbacks, each of them leading to a different kind of sparse stabilizer: a time-varying periodic feedback, a sampled feedback, and a hybrid feedback.

Let us define the three strategies that we will consider.

The absence of continuous feedback stabilizers is a classical matter in control (as in Brockett, 1983) and a classical approach (Samson, 1991; Sontag & Sussmann, 1980) is the introduction of time-varying periodic with respect to time feedback controls (see also Coron, 1992, Rosier & Coron, 1994 or section 11.2 Coron, 2007). In this spirit, we consider a first strategy, as follows. Throughout the article, we denote by e_i the unitary vector in the ith variable.

Strategy 1 (*Sparse Periodic Feedback*). *Fix the sampling time $\tau > 0$ and the final control time $T > 0$. For the initial state $x_0 \in \mathbb{R}^n$, consider the unique trajectory $x(t)$ of (1) with the time-varying feedback control $u(t, x)$ defined as follows:*

- *for each time interval $[(km + i - 1)\tau, (km + i)\tau) \cap [0, T]$ for some $k \in \mathbb{N}$ and $i = 1, \ldots, m$, apply the feedback control*

  $$u(t, x) = -L_{g_i}V(x)e_i;$$

- *for $t \geq T$, apply the zero control $u(t, x) = 0$.*

In our second *sampling* approach, we discretize the time horizon and we apply a fixed control u_i on each interval. Such a control is chosen with a steepest descent approach, by maximizing the instantaneous decrease of V at the beginning of the sampling interval.

Strategy 2 (*Sampled Sparse Feedback*). *Consider the component-wise sparse feedback defined at any $x \in \mathbb{R}^n$ by*

$$u(x) = -L_{g_i}V(x)e_i, \tag{3}$$

where $i \in \{1, \ldots, m\}$ is the smallest integer such that

$$|L_{g_i}V(x)| \geq |L_{g_j}V(x)|, \quad \forall j \neq i. \tag{4}$$

Fix a sampling time $\tau > 0$. Then consider the sampling solution associated with u and the sampling time τ, namely the solution of

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(x(k\tau))g_i(x(t)), \quad t \in [k\tau, (k+1)\tau],$$

with $k \in \mathbb{N}$.

The notion of stabilization associated with sampling solutions is the stabilization in the sample-and-hold sense (see for instance Section 7 Clarke, 2011).

Definition 1.1. Let $U \subset \mathbb{R}^m$, let $F : \mathbb{R}^n \times U \to \mathbb{R}^n$ be continuous and locally Lipschitz in x, uniformly on compact subsets of $\mathbb{R}^n \times U$, with $F(\bar{x}, 0) = 0$. We say that a feedback $u : \mathbb{R}^n \to U$ *stabilizes* the system $\dot{x} = F(x, u(x))$ to $\bar{x}$ *in the sample-and-hold sense* if for every $r > 0$ and $R > 0$ there exists $\tau > 0$ and $T > 0$ depending only on r and R and $C > 0$ depending only on R such that for any $x_0 \in B_R(\bar{x})$ the sampled solution of $\dot{x} = F(x, u(x)), x(0) = x_0$, with sampling time τ satisfies $|x(t)| \leq C,\ \forall t \geq 0$ and $x(t) \in B_r(\bar{x}),\ \forall t \geq T$.

Finally, we consider an hybrid approach based on *hysteresis*: we choose the control component u_i maximizing the instantaneous decrease of V. It is the only active one while it satisfies the lower threshold condition $|L_{g_i}V| > (1-\varepsilon)|L_{g_j}V|$ for any $j \neq i$. When such lower threshold is reached, the control switches to the new control maximizing the instantaneous decrease of V.

Strategy 3 (*Sparse Feedback with Hysteresis*). *Fix $\varepsilon \in (0, 1)$ and apply the following algorithm to define the trajectory $x(t)$ of the system:*

`Initialize:` *$n = 0$ and $t_0 = 0$.*
`While` *$t_n < +\infty$* `apply Step` *n: At time t_n choose $i = 1, \ldots, m$ being the smallest integer such that*

$$|L_{g_i}V(x(t_n))| \geq |L_{g_j}V(x(t_n))|, \tag{5}$$

for every $j \neq i$.

- *If $|L_{g_i}V(x(t_n))| \geq 2t_n^{-1}$, define the switching time t_{n+1} as the infimum $t \in [t_n, +\infty)$ such that the unique solution $y(t)$ of $\dot{y} = f(y) - L_{g_i}V(y)g_i(y)$ with $y(t_n) = x(t_n)$ satisfies one of the following:*

  $$|L_{g_i}V(y(t))| \leq t^{-1} \tag{6}$$

  $$|L_{g_i}V(y(t))| \leq \max_{j \neq i}\{(1-\varepsilon)|L_{g_j}V(y(t))|\}, \tag{7}$$