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Brief paper

Stochastic stability of Markov jump hyperbolic systems with application to traffic flow control[☆]

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ABSTRACT

In this paper, we investigate the stochastic stability of linear hyperbolic conservation laws governed by a finite-state Markov chain. Both system matrices and boundary conditions are subject to the Markov switching. The existence and uniqueness of weak solutions are developed for the stochastic hyperbolic initial–boundary value problem. By means of Lyapunov techniques some sufficient conditions are obtained by seeking the balance between the boundary condition and the transition probability of the Markov process. Particularly, boundary feedback control of a stochastic traffic flow model is developed for the freeway transportation system by integrating the on-ramp metering with the speed limit control.

1. Introduction

Many physical or engineering processes may be represented by the hyperbolic partial differential equations (PDEs) of conservation laws in one space dimension, such as Saint-Venant equation for open channels (de Halleux, Prieur, Coron, d'Andréa Novel, & Bastin, 2003), Euler equation for gas pipes (Gugat, Dick, & Leugering, 2011), and Aw–Rascle equation for road traffic (Aw & Rascle, 2000). In such systems, the system matrices and the boundary conditions can both be subject to abrupt changes in their structures and parameters induced by the external causes or the internal mechanism. For example, in the freeway transportation systems, it can be the phase transition of traffic modes (Colombo, 2003), or the random flux at the boundaries (Haut, Bastin, Coron, & d'Andréa Novel, 2007). In such situations, it is more realistic to model the dynamic behaviors of these processes with switched hyperbolic systems.

Many results have been made for boundary stability of switched hyperbolic systems. In Amin, Hante, and Bayen (2012), the exponential stability is given under arbitrary switching using the propagation of solutions along the characteristics. In Prieur, Girard, and Witrant (2014), using Lyapunov techniques some sufficient conditions are obtained for the exponential stability uniformly. Switching boundary control for semilinear hyperbolic balance equations is considered in Hante, Leugering, and Seidman (2009). In Lamare, Girard, and Prieur (2015), stabilizing switching controllers are developed based on the steepest decent selection of the Lyapunov function.

In this paper, we consider a class of switched hyperbolic systems, named the Markov jump linear hyperbolic (MJLH) systems, in which mode switching is governed by a Markov chain and all modes are linear hyperbolic conservation laws. The boundary stabilization for hyperbolic systems (Coron, d'Andréa Novel, & Bastin, 2007; Li, 1994) and the stochastic stability for the Markov jump linear (MJL) systems of continuous-time (Costa, Fragoso, & Todorov, 2013) or discrete-time cases (Boukas, 2005) have been studied for many years independently. The main contribution of this work is that the boundary stochastic stability for the MJLH systems is firstly obtained by means of Lyapunov techniques. The matrix inequality condition is based on the balance between the boundary condition of the hyperbolic conservation laws and the transition probability of the Markov process.

A second contribution of our work is the application to the boundary control of freeway traffic. Due to the existence of a large number of uncertainties, such as demand variability and capacity decrease, the local freeway traffic may randomly lie in the free-flow mode or in the congestion mode (Boel & Mihaylova, 2006; Sumalee, Zhong, Pan, & Szeto, 2011). Then we develop a two-mode

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MJLH model to represent the quasilinear Aw–Rascle equation and design boundary feedback strategies by integrating the on-ramp metering with the speed limiting control. Theoretical contributions guarantee the stochastic exponential convergence of the MJLH traffic flow model, even with different transition probabilities of the Markov chain.

This paper is organized as follows. The class of MJLH systems and the wellposedness of weak solutions are given in Section 2. In Section 3, the main result on the sufficient conditions of the exponentially mean-square stable are derived for MJLH systems. Numerical computation of the conditions is discussed in Section 4. Finally, in Section 5, as a matter of illustration, an application of boundary feedback control of freeway traffic based on the MJLH traffic flow model is presented.

Notation: $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ are the sets of non-negative reals, n-order vectors and matrices, respectively. The set of diagonal positive matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathcal{D}^n_+$. Given a matrix A, the transpose matrix is denoted as $A^\top$, $\lambda_{max}(A)$, $\rho(A)$ are the largest real parts of all eigenvalues and the spectral radius of A. $A < (\leq) B$ denotes $B - A$ is a positive definite (semi-definite) matrix. Given two real values t_1 and t_2, $t_1 \wedge t_2$ denotes the minimal value between t_1 and t_2. The Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$ and the associated matrix norm is $\|\cdot\|$. Given a function $g : [0, 1] \to \mathbb{R}^n$, its L^2-norm is $\|g\|_{L^2(0,1)} = \sqrt{\int_0^1 |g(x)|^2 dx}$. We call $L^2(0, 1)$ the space of all measurable functions $g(x)$ for which $\|g\|_{L^2(0,1)} < \infty$.

2. Markov jump linear hyperbolic systems

Let $(\Omega, \mathcal{F}, \Pr)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t; t \in \mathbb{R}_+\}$ satisfying the usual hypotheses, that is, a right-continuous filtration augmented by all null sets in the Pr-completion of $\mathcal{F}$.

We consider a homogeneous Markov process $\{\sigma(t); t \in \mathbb{R}_+\}$ adapted to the filtration $\{\mathcal{F}_t; t \in \mathbb{R}_+\}$, with right-continuous trajectories and taking values on the set $\mathcal{S} = \{1, 2, \ldots, N\}$, where N is a positive integer number. The infinitesimal generator $\Pi \in \mathbb{R}^{N\times N}$ of Markov process $\sigma(t)$ is given by

$$\Pr\{\sigma(t+\Delta t) = j|\sigma(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if} \quad i \neq j \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{if} \quad i = j \end{cases} \tag{1}$$

where $\Delta t > 0$ is constant (it is seen as a small time increment) and $o(\cdot)$ is a function satisfying $\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$. Here $\pi_{ij} \geq 0$, for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t + \Delta t$, while

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}.$$

Let $\{\tau_k; k = 0, 1, \ldots\}$ be the successive sojourn times between jumps, then $t_k = \sum_{l=0}^{k-1} \tau_l$, for $k = 1, 2, \ldots$, be the waiting time for the kth jump with $t_0 = 0$.

Stating in mode $\sigma(0) = i$, the process sojourns there for a duration of time that is exponentially distributed with parameter $-\pi_{ii}$. The process then jumps to mode $j \neq i$ with probability $-\frac{\pi_{ij}}{\pi_{ii}}$, and the sojourn time in mode j is exponentially distributed with parameter $-\pi_{jj}$, and so on. We further assume that the Markov process is irreducible. Under this condition, $\sigma(t)$ has a unique stationary probability distribution $\gamma = [\gamma_1 \ldots \gamma_N]^\top$, which can be determined by solving the following linear equation $\gamma^\top \Pi = 0$ subject to $\sum_{j=1}^{N} \gamma_j = 1$ and $\gamma_j > 0$, for all $j \in \mathcal{S}$ (Costa et al., 2013 Definition 2.9).

We consider the following Markov jump linear hyperbolic (MJLH for short) conservation laws of the form

$$\partial_t \xi(x, t) + \Lambda_{\sigma(t)} \partial_x \xi(x, t) = 0, \tag{2}$$

where $t \in \mathbb{R}_+$, $x \in [0, 1]$, $\xi : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}^n$ is the system state, and the Markov process $\sigma(t) : \mathbb{R}_+ \to \mathcal{S}$ is a stochastic switching signal deciding the current operation mode. For all $i \in \mathcal{S}$, the system matrix $\Lambda_i \in \mathbb{R}^{n\times n}$ is a diagonal matrix with non-zero diagonal entries such that

$$\Lambda_i = \mathrm{diag}\{\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i\},$$

with $\lambda_j^i < 0$ for $j \in \{1, \ldots, m_i\}$ and $\lambda_j^i > 0$ for the other $j \in \{m_i + 1, \ldots, n\}$.

According to the sign of each characteristic velocity λ_j^i, $j = \{1, \ldots, n\}$, $i \in \mathcal{S}$, we introduce the notation $\xi_-^i(\cdot) = [\xi_1(\cdot), \ldots, \xi_{m_i}(\cdot)]^\top$, $\xi_+^i(\cdot) = [\xi_{m_i+1}(\cdot), \ldots, \xi_n(\cdot)]^\top$, and thus $\xi = [\xi_-^i, \xi_+^i]^\top$, for all $i \in \mathcal{S}$.

For MJLH system (2), the boundary condition also is a stochastic process, corresponding to the Markov chain $\sigma(t)$, written as

$$\begin{bmatrix} \xi_-^{\sigma(t)}(1, t) \\ \xi_+^{\sigma(t)}(0, t) \end{bmatrix} = G_{\sigma(t)} \begin{bmatrix} \xi_-^{\sigma(t)}(0, t) \\ \xi_+^{\sigma(t)}(1, t) \end{bmatrix}, \tag{3}$$

where G_i is a matrix in $\mathbb{R}^{n\times n}$, $i \in \mathcal{S}$. Let us introduce the matrices G_{--}^i in $\mathbb{R}^{m_i\times m_i}$, G_{-+}^i in $\mathbb{R}^{m_i\times(n-m_i)}$, G_{+-}^i in $\mathbb{R}^{(n-m_i)\times m_i}$, and G_{++}^i in $\mathbb{R}^{(n-m_i)\times(n-m_i)}$ such that $G_i = \begin{bmatrix} G_{--}^i & G_{-+}^i \\ G_{+-}^i & G_{++}^i \end{bmatrix}$.

We consider the initial condition given by

$$\xi(x, 0) = \xi^0(x), \ x \in (0, 1), \tag{4}$$

for a given function $\xi^0(\cdot) \in L^2(0, 1)$ and a initial operation mode $\sigma(0) \in \mathcal{S}$.

For each mode $i \in \mathcal{S}$, as respective hyperbolic equation (2)–(4) holds a sojourn for a duration of time $\{\tau_k; k = 0, 1, \ldots\}$, the existence and uniqueness of solution in the set $C^0([0, \infty), H^1(0, 1)) \cap C^1([0, \infty), L^2(0, 1))$ with initial condition in $L^2(0, 1)$ is quite classical, see e.g. Bastin and Coron (2016, Theorem A.4). Recently, the notion of solutions for an initial–boundary value problem of switched hyperbolic systems has been developed within the usual Lebesgue almost everywhere equivalence class, see e.g. Amin et al. (2012), and Prieur et al. (2014, Proposition 3.1).

We now provide an existence and uniqueness result for the solutions of the MJLH system (2)–(4).

Proposition 1. *The MJLH system (2)–(3) admits a unique solution $\xi = \xi(\cdot, t)$, $t \in \mathbb{R}_+$, such that $\mathrm{E}\left\{\|\xi(\cdot, t)\|_{L^2(0,1)}\right\} < \infty$, for any initial condition $\xi^0 \in L^2(0, 1)$ and any initial operation mode $\sigma(0) \in \mathcal{S}$, where $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation.*

Proof. Recall that almost every sample path of stochastic process $\sigma(t)$, $t \geq 0$, is a right-continuous step function with a finite number of jumps in any finite time interval. Then there exists a sequence $\{t_k; k = 0, 1, \ldots\}$ of stopping times such that $t_0 = 0$, $\lim_{k\to\infty} t_k = \infty$, and $\sigma(t) = \sigma(t_k)$ on $t_k \leq t < t_{k+1}$ for any $k \geq 0$.

We then build iteratively the solution between successive stopping times. Let $T \in \mathbb{R}_+$ be arbitrary, we first consider the MJLH system (2)–(3) on the time interval $t \in [0, t_1 \wedge T]$ which becomes

$$\partial_t \xi(x, t) + \Lambda_{\sigma(0)} \partial_x \xi(x, t) = 0, \tag{5}$$

for all $x \in (0, 1)$ with the boundary condition of the form

$$\begin{bmatrix} \xi_-^{\sigma(0)}(1, t) \\ \xi_+^{\sigma(0)}(0, t) \end{bmatrix} = G_{\sigma(0)} \begin{bmatrix} \xi_-^{\sigma(0)}(0, t) \\ \xi_+^{\sigma(0)}(1, t) \end{bmatrix}, \tag{6}$$

and the initial condition $\xi^0 \in L^2(0, 1)$. For any initial mode $\sigma(0) \in \mathcal{S}$, by Bastin and Coron (2016, Theorem A.4), the initial–boundary