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Brief paper

An optimal control problem for mean-field forward–backward stochastic differential equation with noisy observation ☆

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ABSTRACT

This article is concerned with an optimal control problem derived by mean-field forward–backward stochastic differential equation with noisy observation, where the drift coefficients of the state equation and the observation equation are linear with respect to the state and its expectation. The control problem is different from the existing literature concerning optimal control for mean-field stochastic systems, and has more applications in mathematical finance, e.g., asset–liability management problem with recursive utility, systematic risk model. Using a backward separation method with a decomposition technique, two optimality conditions along with two coupled forward–backward optimal filters are derived. Linear–quadratic optimal control problems for mean-field forward–backward stochastic differential equations are studied.

1. Introduction

1.1. Notation

We denote by $T > 0$ a fixed time horizon, by $\mathbb{R}^m$ the m-dimensional Euclidean space, by $|\cdot|$ (resp. $\langle\cdot,\cdot\rangle$) the norm (resp. scalar product) in a Euclidean space, by $A^\top$ (resp. A^{-1}) the transposition (resp. reverse) of A, by S^m the set of symmetric $m\times m$ matrices with real elements, by f_x the partial derivative of f with respect to x, and by C a positive constant, which can be different from line to line. Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{0\le t\le T},\mathbb{P})$ be a complete filtered probability space, on which are given an $\mathscr{F}_t$-adapted standard Brownian motion $(w_t,\tilde{w}_t)$ with values in $\mathbb{R}^{r+\tilde{r}}$ and a Gaussian random variable ξ with mean μ_0 and covariance matrix σ_0. $(w,\tilde{w})$ is independent of ξ. If $A\in S^m$ is positive (semi) definite, we write $A>(\ge)0$. If $x:[0,T]\to\mathbb{R}^m$ is uniformly bounded, we write $x\in\mathscr{L}^\infty(0,T;\mathbb{R}^m)$. If $x:\Omega\to\mathbb{R}^m$ is an $\mathscr{F}_T$-measurable, square-integrable random variable, we write $x\in\mathscr{L}^2_{\mathscr{F}}(\mathbb{R}^m)$. If $x:[0,T]\times\Omega\to\mathbb{R}^m$ is an $\mathscr{F}_t$-adapted, square-integrable process, we write $x\in\mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$. We also adopt similar notations for other processes, Euclidean spaces and filtrations.

1.2. Motivation

Now consider an asset–liability management problem of a firm. Let the dimension $n=k=r=\tilde{r}=1$. Denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$, by v_t the control strategy of the firm, by x_t^v the cash-balance, and by $\bar{l}_t^v$ the liability process. Norberg (1999) described the liability process by a Brownian motion with drift. The model, however, is not just the one we want. In fact, it is possible that the control strategy and the mean of the cash-balance can influence the liability process, due to the complexity of the financial market and the risk aversion behavior of the firm. Such an example can be found in Huang, Wang, and Wu (2010), where the liability process depends on a control strategy (e.g., capital injection or withdrawal) of the firm. Along this line, we proceed to improve the liability process here. Suppose that $\bar{l}_t^v$ satisfies a linear stochastic differential equation (SDE, in short)

$$-d\bar{l}_t^v=(\bar{a}_t\mathbb{E}x_t^v+b_tv_t+\bar{b}_t)dt+c_tdw_t.$$

Here $\bar{a},b,\bar{b},c,a,f,g$ and h are deterministic and uniformly bounded. $\bar{b}_t$ and c_t denote the liability rate and the volatility coefficient, respectively. Suppose that the firm owns an initial investment ξ, and only invests in a money account with the compounded interest rate a_t. Then the cash-balance of the firm is

$$x_t^v=e^{\int_0^t a_sds}\left(\xi-\int_0^t e^{-\int_0^s a_rdr}d\bar{l}_s^v\right).$$

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It follows from Itô's formula that

$$\begin{cases} dx_t^v = \left(a_t x_t^v + \bar{a}_t \mathbb{E}x_t^v + b_t v_t + \bar{b}_t\right) dt + c_t dw_t, \\ x_0^v = \xi. \end{cases}$$

Note that, if $b_t = 1$, $\bar{b}_t = 0$, $a_t = -\bar{a}_t =$ const. and $c_t =$ const., then the cash-balance equation is just the systematic risk model of inter-bank borrowing and lending introduced in Carmona, Fouque, and Sun (2015). Besides the systematic risk model, the equation can also be reduced to an air conditioning control model in energy-efficient buildings. See, e.g., Example 2 in Djehiche, Tembine, and Tempone (2015) for more details.

Due to the discreteness of account information, it is possible for the firm to partially observe the cash-balance by the stock price

$$\begin{cases} dS_t^v = S_t^v \left[\left(f_t x_t^v + g_t + \frac{1}{2} h_t^2 \right) dt + h_t d\tilde{w}_t \right], \\ S_0^v = 1. \end{cases}$$

Set $Y_t^v = \log S_t^v$. It holds that Y^v is governed by

$$\begin{cases} dY_t^v = \left(f_t x_t^v + g_t\right) dt + h_t d\tilde{w}_t, \\ Y_0^v = 0. \end{cases}$$

Suppose that the firm has triple performance objectives. The first two ones are to minimize the total cost of v over $[0, T]$ and to minimize the risk of x_T^v. Assume that the risk is measured by $\mathbb{E}\left[(x_T^v - \mathbb{E}x_T^v)^2\right]$. The third one is to maximize the utility y_t^v resulting from v. Without loss of generality, define $y_t^v = \mathbb{E}\left[x_T^v + \int_t^T G(s, y_s^v, v_s)ds | \mathscr{F}_t\right]$, where G is Lipschitz continuous with respect to (y, v), and $G(s, 0, 0) \in \mathscr{L}^2_{\mathscr{F}}(0, T; \mathbb{R})$ for $0 \le s \le T$. We emphasize that the current utility y_t^v depends not only on the instantaneous control v_t, but also on the future utility and control (y_s^v, v_s), $t \le s \le T$. This shows the difference between the utility y^v and the standard additive utility, and hence, y^v is called as a stochastic differential recursive utility. Then the asset–liability management problem with recursive utility is stated as follows.

Problem (AL). Find a $\sigma\{Y_s^v; 0 \le s \le t\}$-adapted and square-integrable process v_t such that

$$J[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T B_t v_t^2 dt + H(x_T^v - \mathbb{E}x_T^v)^2 - 2N y_0^v\right]$$

is minimized. Here $B > 0$ and B^{-1} are deterministic and uniformly bounded. H and N are non-negative constants. y_0^v is the value of y_t^v at time 0.

According to El Karoui, Peng, and Quenez (1997), the recursive utility y_t^v admits the backward SDE (BSDE, in short)

$$\begin{cases} -dy_t^v = G(t, y_t^v, v_t)dt - z_t^v dw_t - \tilde{z}_t^v d\tilde{w}_t, \\ y_T^v = x_T^v. \end{cases}$$

With the BSDE, Problem (AL) can be rewritten as an optimal control problem derived by forward–backward SDE (FBSDE, in short) with noisy observation.

1.3. Problem statement

Motivated by the examples, we study an optimal control problem for FBSDE with noisy observation. Consider a controlled FBSDE

$$\begin{cases} dx_t^v = \left(a_t x_t^v + \bar{a}_t \mathbb{E}x_t^v + b(t, v_t)\right) dt + c_t dw_t, \\ -dy_t^v = (\alpha_t x_t^v + \bar{\alpha}_t \mathbb{E}x_t^v + \beta_t y_t^v + \bar{\beta}_t \mathbb{E}y_t^v + \gamma_t z_t^v \\ \quad + \bar{\gamma}_t \mathbb{E}z_t^v + \tilde{\gamma}_t \tilde{z}_t^v + \bar{\tilde{\gamma}}_t \mathbb{E}\tilde{z}_t^v + \psi(t, v_t))dt \\ \quad - z_t^v dw_t - \tilde{z}_t^v d\tilde{w}_t, \\ x_0^v = \xi, \quad y_T^v = \rho x_T^v + \bar{\rho}\, \mathbb{E}x_T^v, \end{cases}$$

where $(x^v, y^v, z^v, \tilde{z}^v)$ is the state, v is the control, and $(w, \tilde{w})$ is the Brownian motion. Since the mean of the state influences the state equation, we call the equation a mean-field FBSDE, or a McKean–Vlasov FBSDE. Assume that $(x^v, y^v, z^v, \tilde{z}^v)$ is partially observed through

$$\begin{cases} dY_t^v = \left(f_t x_t^v + \bar{f}_t \mathbb{E}x_t^v + g(t, v_t)\right) dt + h_t d\tilde{w}_t, \\ Y_0^v = 0. \end{cases}$$

The cost functional is

$$J[v] = \mathbb{E}\left[\int_0^T l(t, x_t^v, \mathbb{E}x_t^v, v_t)dt + \phi(x_T^v, \mathbb{E}x_T^v) + \varphi(y_0^v)\right].$$

Here v_t is required to be $\sigma\{Y_s^v; 0 \le s \le t\}$-adapted and to satisfy $\mathbb{E}\sup_{0 \le t \le T} |v_t|^2 < +\infty$. a, $\bar{a}$, b, c, α, $\bar{\alpha}$, β, $\bar{\beta}$, γ, $\bar{\gamma}$, $\tilde{\gamma}$, $\bar{\tilde{\gamma}}$, ψ, ρ, $\bar{\rho}$, f, $\bar{f}$, g, h, l, ϕ and φ will be specified in Section 2. Our problem is to select an admissible control v to minimize $J[v]$. We denote the mean-field type control problem by Problem (MFC).

To solve Problem (MFC), it is natural to use dynamic programming and maximum principle. The dynamic programming, however, does not hold even if the BSDE and the observation equation are not present, mainly due to the inclusion of the mean of the state, which leads to the time inconsistency. We instead study the maximum principle for optimality of Problem (MFC).

1.4. Briefly historical retrospect and contribution of this paper

Mean-field theory provides an effective tool for investigating the collective behaviors arising from individuals' mutual interactions in various different fields, say, finance, game, engineering. Since the independent introduction by Huang, Caines, and Malhamé (2006, 2007) and Lasry and Lions (2007), the mean-field theory has attracted more attention. See, e.g., Buckdahn, Djehiche, and Li (2011), Djehiche et al. (2015), Elliott, Li, and Ni (2013), Hu, Nualart, and Zhou (2014), Huang, Li, and Yong (2015), Meyer-Brandis, Øksendal, and Zhou (2012), Ni, Elliott, and Li (2015), Shen, Meng, and Shi (2014), Wang, Zhang, and Zhang (2014) and Yong (2013) for mean-field control of SDE; Bensoussan, Sung, Yam, and Yung (2016), Carmona, Delarue, and Lachapelle (2013), Carmona et al. (2015) and Tembine, Zhu, and Basar (2014) for mean-field game of SDE. Both mean-field control and mean-field game lead to mean-field FBSDE. Buckdahn, Djehiche, Li, and Peng (2009) studied the well-posedness of a decoupled mean-field FBSDE. Bensoussan, Yam, and Zhang (2015) and Carmona and Delarue (2015) extended Buckdahn et al. (2009) to the case of fully coupled mean-field FBSDE. See also Ahmed and Ding (2001) and Bensoussan, Frehse, and Yam (2013) for other developments of mean-field theory.

Mean-field FBSDE is a well-defined dynamic system, it is very appealing to study control problems for mean-field FBSDEs. To our knowledge, there is only a few literature on this topic. For example, Li and Liu (2014) investigated an optimal control problem for fully coupled mean-field FBSDE. Hafayed, Tabet, and Boukaf (2015) obtained a maximum principle for mean-field FBSDE with jump. In this paper, we focus on studying a controlled mean-field FBSDE with noisy observation, i.e., Problem (MFC). This problem has three new features as follows. (1) The drift coefficient of the observation equation is linear with respect to the state and its expectation, and the observation noise is correlated with the state noise. (2) The classical separation principle does not work, mainly due to the fact that the mean square error of filtering of BSDE depends on the control in general. (3) The state equation involves the mean of the state, and thus, Problem (MFC) cannot be transformed into a standard control problem for FBSDE in a large state space. See, e.g., Wang, Xiao, and Xing (2015) for an illustrative example. Note that there is a circular dependence between v and Y^v, which results in the unavailability of classical variation. Due to the first feature above, the usual approach such as Girsanov's measure transformation Wang, Wu, and Xiong (2013) cannot be used to decouple the