Automatica 85 (2017) 113–121

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

On computing the distance to stability for matrices using linear dissipative Hamiltonian systems☆

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ARTICLE INFO

Article history:
Received 4 November 2016
Received in revised form 8 February 2017
Accepted 18 June 2017
Available online 19 August 2017

Keywords:
Dissipative Hamiltonian systems
Distance to stability
Convex optimization

ABSTRACT

In this paper, we consider the problem of computing the nearest stable matrix to an unstable one. We propose new algorithms to solve this problem based on a reformulation using linear dissipative Hamiltonian systems: we show that a matrix A is stable if and only if it can be written as $A = (J - R)Q$, where $J = -J^T$, $R \succeq 0$ and $Q \succ 0$ (that is, R is positive semidefinite and Q is positive definite). This reformulation results in an equivalent optimization problem with a simple convex feasible set. We propose three strategies to solve the problem in variables (J, R, Q): (i) a block coordinate descent method, (ii) a projected gradient descent method, and (iii) a fast gradient method inspired from smooth convex optimization. These methods require $\mathcal{O}(n^3)$ operations per iteration, where n is the size of A. We show the effectiveness of the fast gradient method compared to the other approaches and to several state-of-the-art algorithms.

1. Introduction

In this paper, we focus on the continuous linear time invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, x is the state vector and u is the input vector. Such a system is stable if all eigenvalues of A are in the *closed* left half of the complex plane and all eigenvalues on the imaginary axis are semisimple. Therefore the stability solely depends on A, and the matrix B that weights the inputs can be ignored to study stability.

For a given unstable matrix A, the problem of finding the smallest perturbation that stabilizes A, or, equivalently finding the nearest stable matrix X to A is an important problem (Orbandexivry, Nesterov, & Van Dooren, 2013). More precisely, we consider the following problem. For a given unstable matrix A, compute

$$\inf_{X \in \mathbb{S}^{n,n}} \|A - X\|_F^2, \tag{1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix and $\mathbb{S}^{n,n}$ is the set of all stable matrices of size $n \times n$. This problem occurs for example in system identification where one needs to identify a stable system from observations (Orbandexivry et al., 2013).

The converse of problem (1) is the stability radius problem, where a stable matrix A is given and one looks for the smallest perturbation that moves an eigenvalue outside the stability region (Byers, 1988; Hinrichsen & Pritchard, 1986). Both problems are nontrivial because even a small perturbation on the coefficients of the matrix may move the eigenvalues in any direction and the perturbed matrix may well have eigenvalues that are far from those of A (Orbandexivry et al., 2013). However, the nearest stable matrix problem appears to be more difficult since it requires to push all eigenvalues from the unstability region into the stability region while the stability radius problem only requires to move a single eigenvalue on the boundary of the stability region.

The various distance problems for matrices have been a topic of research in the numerical linear algebra community, for example, matrix nearness problems (Higham, 1988b), the structured singular value problem (Packard & Doyle, 1993), the robust stability problem (Zhou, 2011), the distance to bounded realness for Hamiltonian matrices (Alam, Bora, Karow, Mehrmann, & Moro, 2011), and the nearest defective matrix (Wilkinson, 1984).

Another related problem is to find the closest stable polynomial to a given unstable one. This was addressed by Moses and Liu (1991), where an algorithm using the alternating projection approach in Schur parameter space was developed. But the technique developed in Moses and Liu (1991) is limited and cannot be applied to other types of systems. In Burke, Henrion, Lewis, and Overton (2006b), authors stabilize fixed order controllers using nonsmooth, nonconvex optimization. A MATLAB toolbox called HIFOO (H_∞

☆ The authors acknowledge the support of the ERC (starting grant no 679515). Nicolas Gillis also acknowledges the support of the F.R.S.-FNRS (incentive grant for scientific research no F.4501.16). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Delin Chu under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2017.07.047
0005-1098/

fixed order optimization) was designed to solve fixed order stabilization and local optimization problems (Burke, Henrion, Lewis, & Overton, 2006a). In D'haene, Pintelon, and Vandersteen (2006), authors stabilize transfer functions using a two step iterative procedure that guaranteed stable transfer function models from noisy data.

We note that in the literature, a stable matrix is sometimes considered to satisfy $\mathrm{Re}(\lambda) < 0$ for all its eigenvalues λ; see, e.g., Byers (1988), Hinrichsen and Pritchard (1986) and Orbandexivry et al. (2013). To avoid the confusion, we call such matrices asymptotically stable. The set of all asymptotically stable matrices is open. This follows from the fact that the eigenvalues of a matrix depend continuously on its entries (Ostrowski, 1960). However, the set $\mathbb{S}^{n,n}$ is neither open nor closed, because $A_\epsilon \notin \mathbb{S}^{n,n}$ for $\epsilon > 0$, but $A \in \mathbb{S}^{n,n}$, where

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}}_{=:A_\epsilon} \to \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}}_{=:A},$$

and $B_\delta \in \mathbb{S}^{n,n}$ for $\delta < 0$, but $B \notin \mathbb{S}^{n,n}$, where

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \delta & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}}_{=:B_\delta} \to \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}}_{=:B}.$$

Further, the set $\mathbb{S}^{n,n}$ of stable matrices in (1) is highly nonconvex (Orbandexivry et al., 2013) and therefore it is in general difficult to compute a global optimal solution to problem (1).

Our work is mainly motivated by the work in Orbandexivry et al. (2013), where a nearby stable approximation X of a given unstable system A is constructed by means of successive convex approximations of the set of stable systems. Our principle strategy for computing a nearby stable approximation to a given unstable matrix is to reformulate the highly nonconvex optimization problem (1) into an equivalent (non-convex) optimization problem with a convex feasible region onto which points can be projected relatively easily. We aim to provide in many cases better approximations than the one obtained with the code from Orbandexivry et al. (2013) by using the concept of linear dissipative Hamiltonian systems.

Notation: In the following, we denote $A \succ 0$ and $A \succeq 0$ if A is symmetric positive definite or symmetric positive semidefinite, respectively. The set $\Lambda(A)$ denotes the set of all eigenvalues of A.

1.1. Dissipative Hamiltonian systems

A dissipative Hamiltonian (DH) system in the linear time invariant case can be expressed as

$$\dot{x} = (J - R)Qx,$$

where the function $x \to x^T Qx$ with $Q = Q^T \in \mathbb{R}^{n,n}$ positive definite describes the energy of the system, $J = -J^T \in \mathbb{R}^{n,n}$ is the structure matrix that describes flux among energy storage elements, and $R \in \mathbb{R}^{n,n}$ with $R = R^T \succeq 0$ is the dissipation matrix that describes energy dissipation in the system. DH systems are special cases of port-Hamiltonian systems, which recently have received a lot attention in energy based modeling; see, e.g., Golo, van der Schaft, Breedveld, and Maschke (2003), van der Schaft (2006) and van der Schaft and Maschke (2013). An important property of DH systems is that they are stable, i.e., all eigenvalues of matrix $A = (J - R)Q$ are in the closed left half of the complex plane and all eigenvalues on the imaginary axis are semisimple. This follows from the fact that Q is symmetric positive definite. Indeed, for any nonzero vector z one has

$$\begin{aligned} \mathrm{Re}\left(z^*(Q^{\frac{1}{2}}AQ^{-\frac{1}{2}})z\right) &= \mathrm{Re}\left(z^*(Q^{\frac{1}{2}}JQ^{\frac{1}{2}} - Q^{\frac{1}{2}}RQ^{\frac{1}{2}})z\right) \\ &= -z^*Q^{\frac{1}{2}}RQ^{\frac{1}{2}}z \le 0, \end{aligned}$$

since R is positive semidefinite, where $*$ stands for the complex conjugate transpose of a matrix or a vector. The semisimplicity of the purely imaginary eigenvalues of $(J - R)Q$ follows from Mehl, Mehrmann, and Sharma (2016, Lemma 3.1). The various structured distances of a DH system from the region of asymptotic stability have recently been studied in Mehl et al. (2016) for the complex case and in Mehl, Mehrmann, and Sharma (2017) for the real case.

This paper is organized as follows. In Section 2, we reformulate the nearest stable matrix problem using the notion of DH matrices. We also provide several theoretical results necessary to obtain our reformulation. In Section 3, three algorithms are proposed to solve the reformulation. In Section 4, we present numerical experiments that illustrate the performance of our algorithms and compare the results with several state-of-the-art algorithms.

2. DH framework for checking stability

In this section, we present a new framework based on dissipative Hamiltonian systems to attack the nearest stable matrix problem (1). Our main idea is to reformulate the nonconvex optimization problem (1) into an equivalent optimization problem with a relatively simple convex feasible set. In order to do this, let us define a DH matrix.

Definition 1. A matrix $A \in \mathbb{R}^{n,n}$ is said to be a *DH* matrix if $A = (J - R)Q$ for some $J, R, Q \in \mathbb{R}^{n,n}$ such that $J^T = -J$, $R \succeq 0$ and $Q \succ 0$.

Clearly from the previous section every DH matrix is stable. In our terminology, (Beattie, Mehrmann, & Xu, 2015 Corollary 2) implies that every stable matrix A is similar to a DH matrix, i.e., there exists T nonsingular such that $T^{-1}AT = (J-R)Q$ for some $J^T = -J$, $R \succeq 0$ and $Q \succ 0$. In fact we prove something stronger: a stable matrix itself is a DH matrix, as shown in the following lemma.

Lemma 2. *Every stable matrix is a DH matrix.*

Proof. Let A be stable. By Lyapunov's theorem (Lancaster & Tismenetsky, 1985), there exists $P \succ 0$ such that

$$AP + PA^T \preceq 0. \tag{2}$$