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A numerical approach to optimal coherent quantum LQG controller design using gradient descent☆

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ABSTRACT

This paper is concerned with coherent quantum linear quadratic Gaussian (CQLQG) control. The problem is to find a stabilizing measurement-free quantum controller for a quantum plant so as to minimize a mean square cost for the fully quantum closed-loop system. The plant and controller are open quantum systems interconnected through bosonic quantum fields. In comparison with the observation–actuation structure of classical controllers, coherent quantum feedback is less invasive to the quantum dynamics. The plant and controller variables satisfy the canonical commutation relations (CCRs) of a quantum harmonic oscillator and are governed by linear quantum stochastic differential equations (QSDEs). In order to correspond to such oscillators, these QSDEs must satisfy physical realizability (PR) conditions in the form of quadratic constraints on the state-space matrices, reflecting the CCR preservation in time. The symmetry of the problem is taken into account by introducing equivalence classes of coherent quantum controllers generated by symplectic similarity transformations. We discuss a modified gradient flow, which is concerned with norm-balanced realizations of controllers. A line-search gradient descent algorithm with adaptive stepsize selection is proposed for the numerical solution of the CQLQG control problem. The algorithm finds a local minimum of the LQG cost over the parameters of the Hamiltonian and coupling operators of a stabilizing coherent quantum controller, thus taking the PR constraints into account. A convergence analysis of the algorithm is presented. Numerical examples of designing locally optimal CQLQG controllers are provided in order to demonstrate the algorithm performance.

1. Introduction

Coherent quantum feedback control (Lloyd, 2000; Mabuchi, 2008) is a quantum control paradigm which is aimed at achieving given performance specifications for quantum systems, such as internal stability and optimization of a cost functional. Such systems arise naturally in quantum physics (Holevo, 2003) and its engineering applications (for example, nanotechnology and quantum optics (Gardiner & Zoller, 2004)). The dynamic variables of quantum systems are (usually noncommuting) operators on an underlying Hilbert space which evolve according to the laws of quantum mechanics (Merzbacher, 1998). The quantum dynamics are particularly sensitive to interaction with classical devices over the course of quantum measurement, as reflected in the projection postulate of quantum mechanics. In order to overcome this issue, coherent quantum control employs the idea of direct interconnection of quantum plants (that is, the quantum systems to be controlled) with other quantum systems playing the role of controllers, possibly mediated by light fields. Unlike the traditional observation–actuation control loop, this fully quantum measurement-free feedback avoids the loss of quantum information resulting from a conversion to classical signals.

While some of the potential applications of coherent quantum feedback control involve nonlinear dynamics and/or non-Gaussian noises, the linear setting is an important starting point for study and presents advantages in terms of analytic and computational tractability. Quantum-optical components, such as optical cavities, beam splitters and phase shifters, make it possible to implement coherent quantum feedback governed by Markovian linear quantum stochastic differential equations (QSDEs) (Parthasarathy, 1992; Petersen, 2016), provided the latter are physically realizable (PR) as open quantum harmonic oscillators (Edwards & Belavkin,

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2005; Gardiner & Zoller, 2004). The resulting PR conditions (James, Nurdin, & Petersen, 2008; Shaiju & Petersen, 2012; Sichani & Petersen, 2016) are organized as quadratic constraints on the coefficients of the QSDEs. The PR constraints for the state-space matrices of a coherent quantum controller complicate the solution of quantum counterparts to the classical Linear Quadratic Gaussian (LQG) and $\mathscr{H}_\infty$ control problems.

The Coherent Quantum LQG (CQLQG) problem is one of the fundamental problems arising in linear coherent quantum control theory (Petersen, 2016). The analogy between this problem and its classical counterpart in the linear control theory can be traced back to the connections between classical and quantum probability frameworks. The CQLQG control problem (Nurdin, James, & Petersen, 2009) seeks for a stabilizing PR quantum controller so as to minimize an infinite-horizon mean square cost functional for the fully quantum closed-loop system. In physically relevant applications, problems such as cooling of optomechanical resonators can be formulated in the framework of the CQLQG control problem (Hamerly & Mabuchi, 2013). The CQLQG control problem is a constrained optimization problem for the steady-state quantum covariance matrix of the plant-controller system satisfying an algebraic Lyapunov equation (ALE). A numerical procedure for finding *suboptimal* controllers for this problem was proposed in Nurdin, James, and Petersen, (2009), and algebraic equations for the *optimal* CQLQG controller were obtained in Vladimirov and Petersen (2013a). We also mention that coherent quantum LQG control settings were considered in Maalouf and Petersen (2009) for a class of quantum systems (with annihilation operators only), in the context of evolutionary optimization for entanglement control (Harno & Petersen, 2015), and also for different scenarios of plant-controller coupling in Zhang and James (2011). Despite the previous results, the CQLQG control problem does not lend itself to an "elegant" solution (for example, in the form of decoupled Riccati equations as in the classical case (Kwakernaak & Sivan, 1972)) and remains a subject of research. Since the main difficulties are caused by the coupling between the ALEs for the state-space matrices of the optimal controller due to the PR constraints, Vladimirov and Petersen (2013b) offered an alternative approach based on moving the "burden" of the constraints to the Lagrange multipliers for a coherent quantum filtering problem (Miao & James, 2012) which is a simplified feedback-free version of the control problem.

In the present paper, we develop an algorithm for the numerical solution of the CQLQG control problem by using a line-search (gradient descent) method and the Hamiltonian parameterization of PR quantum controllers (Vladimirov & Petersen, 2013a). This parameterization is a different technique to handle the PR constraints by reformulating the CQLQG control problem in an unconstrained fashion. More precisely, the optimal solution is sought in the class of stabilizing PR controllers whose state-space matrices are parameterized in terms of the free Hamiltonian and coupling operators of an open quantum harmonic oscillator (Edwards & Belavkin, 2005). We obtain ordinary differential equations (ODEs) for the gradient descent in the Hilbert space of these matrix-valued parameters of coherent quantum controllers. In accordance with the PR conditions, the CQLQG control problem has a special type of *symmetry* which makes it invariant under symplectic similarity transformations of the controller variables (Vladimirov & Petersen, 2011, 2013a). We take this symmetry into account and consider equivalence classes of state-space representations of coherent quantum controllers. We also propose a modified gradient flow, which is concerned with norm-balanced realizations of such controllers and resembles the steepest descent with respect to a different Riemannian metric (Absil, Mahony, & Sepulchre, 2008). For this purpose, we combine the Fréchet and Gâteaux differentiation with differential geometric tools (such as Lie groups (Olver, 1993) and tangent spaces) and related algebraic techniques (Bernstein & Haddad, 1989; Magnus, 1988; Skelton, Iwasaki, & Grigoriadis, 1998; Vladimirov & Petersen, 2012, 2013a) to employ the analytic structure of the LQG cost as a composite function of the matrix-valued variables, whose computation involves ALEs.

A useful feature of the gradient descent approach to the CQLQG control problem is that, at intermediate steps, it produces stabilizing PR quantum controllers which can be regarded as gradually improving suboptimal solutions of the problem, and a locally optimal solution (if it exists) is achieved asymptotically by moving along anti-gradient directions with a suitable choice of stepsizes. To this end, we provide an algorithm for adaptive stepsize selection for each iteration based on the second-order Gâteaux (directional) derivative of the LQG cost along the gradient. However, the proposed gradient descent algorithm for the CQLQG control problem requires for its initialization a *stabilizing* PR quantum controller. Finding such a controller for an arbitrary given quantum plant is a nontrivial open problem which has recently been considered in the frequency domain (Sichani, Petersen, & Vladimirov, 2015). Because of the lack of a systematic solution for this *quantum stabilization problem*, the present algorithm is initialized at a stabilizing PR quantum controller which is obtained by a random search in the space defined by the Hamiltonian parameterization of PR controllers. Although a random search of an admissible starting point is acceptable for low-dimensional problems, the development of a more systematic approach to this issue is a subject of future research.

The paper is organized as follows. Section 2 outlines the principal notation. Section 3 specifies the quantum closed-loop system being considered. We also revisit the PR conditions for linear quantum systems in this section. Section 4 formulates the CQLQG control problem. Section 5 specifies the gradient flow for finding local minima in this problem. Section 6 presents equivalent realizations and norm-balanced realizations of coherent quantum controllers. More specifically, Section 6.1 defines equivalence classes generated by symplectic similarity transformations of coherent quantum controllers. These are used in Section 6.2 which is concerned with norm-balanced realizations of coherent quantum controllers and proposes a modified gradient flow. Section 7 describes an algorithmic implementation of the gradient descent method with an adaptive line search. Section 7.4 discusses the convergence of this algorithm. Section 8 provides numerical examples of designing locally optimal CQLQG controllers. Section 9 gives concluding remarks. Appendices A–H provide proofs of lemmas and theorems along with subsidiary material.

2. Notation

Vectors are assumed to be organized as columns unless specified otherwise, and the transpose $(\cdot)^{\mathrm{T}}$ acts on matrices with operator-valued entries as if the latter were scalars. For a vector X of operators $X_1, \ldots, X_r$ and a vector Y of operators $Y_1, \ldots, Y_s$, the commutator matrix $[X, Y^{\mathrm{T}}] := XY^{\mathrm{T}} - (YX^{\mathrm{T}})^{\mathrm{T}}$ is an $(r \times s)$-matrix whose (j, k)th entry is the commutator $[X_j, Y_k] := X_jY_k - Y_kX_j$ of the operators X_j and Y_k. Furthermore, $(\cdot)^\dagger := ((\cdot)^\#)^{\mathrm{T}}$ denotes the transpose of the entry-wise operator adjoint $(\cdot)^\#$. When it is applied to complex matrices, $(\cdot)^\dagger$ reduces to the complex conjugate transpose $(\cdot)^* := (\overline{(\cdot)})^{\mathrm{T}}$. Denoted by $\mathrm{sym}(\cdot) := \frac{(\cdot)+(\cdot)^{\mathrm{T}}}{2}$ and $\mathrm{asym}(\cdot) := \frac{(\cdot)-(\cdot)^{\mathrm{T}}}{2}$ are the symmetrizer and antisymmetrizer of matrices. Also, we denote by $\mathbb{S}_r$, $\mathbb{A}_r$ and $\mathbb{H}_r := \mathbb{S}_r + i\mathbb{A}_r$ the subspaces of real symmetric, real antisymmetric and complex Hermitian matrices of order r, respectively, with $i := \sqrt{-1}$ the imaginary unit. Denoted by $\mathbf{J} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a matrix which spans the space $\mathbb{A}_2$. Furthermore, I_r denotes the identity matrix of order r, positive (semi-) definiteness of matrices is denoted by $(\succcurlyeq)\ \succ$, and $\otimes$ is the tensor product of spaces or operators (in particular, the Kronecker product of