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Brief paper

Distributed Nash equilibrium seeking for aggregative games with coupled constraints☆

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ABSTRACT

In this paper, we study a distributed continuous-time design for aggregative games with coupled constraints in order to seek the generalized Nash equilibrium by a group of agents via simple local information exchange. To solve the problem, we propose a distributed algorithm based on projected dynamics and non-smooth tracking dynamics, even for the case when the interaction topology of the multi-agent network is time-varying. Moreover, we prove the convergence of the non-smooth algorithm for the distributed game by taking advantage of its special structure and also combining the techniques of the variational inequality and Lyapunov function.

1. Introduction

The seek of generalized Nash equilibria for non-cooperative games with coupled constraints has been widely investigated due to various applications in natural/social science and engineering (such as telecommunication power allocation and cloud computation Ardagna, Panicucci, & Passacantando, 2013; Pang, Scutari, Facchinei, & Wang, 2008). Significant theoretic and algorithmic achievement has been done, referring to Pavel (2007), Altman and Solan (2009), Arslan, Demirkol, and Yueksel (2015) and Facchinei and Kanzow (2010).

Distributed equilibrium seeking algorithms guide a group of players or agents to cooperatively achieve the Nash equilibrium (NE), based on players' local information and information exchange between their neighbors in a network. The NE seeking may be viewed as an extension of distributed optimization problems, which have been widely studied recently (see Kia, Cortés, & Martínez, 2015; Nedić & Ozdaglar, 2009; Shi, Johansson, & Hong, 2013), and on the other hand, distributed optimization problems can be handled with a game-theoretic approach (Li & Marden, 2013). In fact, in the study of complicated behaviors of strategic-interacted players in large-scale networks, it is quite natural to investigate game theory in a distributed way. For example, distributed convergence to NE of zero-sum games over two subnetworks was obtained in Lou, Hong, Xie, Shi, and Johansson (2016). Moreover, a distributed fictitious play algorithm was proposed in Swenson, Kar, and Xavier (2015), while a gossip-based approach was employed for seeking an NE of noncooperative games in Salehisadaghiani and Pavel (2016).

Aggregative games have become an important type of game since the well-known Cournot model was proposed, and have recently been studied in the literature, referring to (Cornes & Hartley, 2012; Jensen, 2010), for its broad application in public environmental models (Cornes, 2016), congestion control of communication networks (Barrera & Garcia, 2015), and demand response management of power systems (Ye & Hu, 2017). Usually, linear aggregation functions and quadratic cost functions in such games were considered, for example, in Paccagnan, Gentile, Parise, Kamgarpour, and Lygeros (2016), Parise, Gentile, Grammatico and Lygeros (2015) and Ye and Hu (2017). Also, a recent result was given for distributed discrete-time algorithms to seek the NE of an aggregative game with time-varying topologies in Koshal, Nedić and Shanbhag (2016).

The objective of this paper is to develop a novel distributed continuous-time algorithm for nonlinear aggregative games with linear coupled constraints and time-varying topologies. In recent years, continuous-time algorithms for distributed optimization become more and more popular (Kia et al., 2015; Shi et al., 2013;

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Yi, Hong, & Liu, 2016), partially because they may be easily implemented in continuous-time or hybrid physical systems. However, ideas and approaches for continuous-time design may not be the same as those for the discrete-time one. Thanks to various well-developed continuous-time methods, distributed continuous-time algorithms or protocols keep being constructed, but the (convergence) conditions may be different from those in discrete-time cases.

In our problem setup, every player tries to optimize its local cost function by updating its local decision variable. The cost function depends on not only the local variable but also a nonlinear aggregation. Moreover, feasible decision variables of players are coupled by linear constraints. Existing distributed algorithms for aggregative games (Koshal et al., 2016; Ye & Hu, 2017) cannot solve our problems since they did not consider coupled constraints. The contribution of this paper can be summarized as follows:

- The aggregative game model in this paper generalizes the previous ones in Paccagnan et al. (2016) and Ye and Hu (2017) by allowing nonlinear aggregation term and non-quadratic cost functions, and also those in Koshal et al. (2016) by considering coupled constraints. In addition, the considered game can be non-potential.
- Inspired from distributed average tracking dynamics and projected primal–dual dynamics, we take advantage of continuous-time techniques to solve the distributed problem. With the new idea, our algorithm is described as a *non-smooth* multi-agent system with two interconnected dynamics: a projected gradient one for the equilibrium seeking, and a consensus one for the synchronization of the aggregation and the dual variables. In addition, our algorithm need not solve the best response subproblems, different from those in Parise et al. (2015), and can keep private some information about the cost functions, local decisions, and constraint coefficients.
- We provide a method to prove the correctness and convergence of the continuous-time algorithm by combining the techniques from variational inequality theory and Lyapunov stability theory.

Notations: Denote $\mathbb{R}^n$ as the n-dimensional real vector space; denote $\mathbf{1}_n = (1, \ldots, 1)^T \in \mathbb{R}^n$, and $\mathbf{0}_n = (0, \ldots, 0)^T \in \mathbb{R}^n$. Denote $col(x_1, \ldots, x_n) = (x_1^T, \ldots, x_n^T)^T$ as the column vector stacked with column vectors $x_1, \ldots, x_n$, $\|\cdot\|$ as the Euclidean norm, and $I_n \in \mathbb{R}^{n\times n}$ as the identity matrix. Denote ∇f as the gradient vector of a function f and $\mathcal{J}F$ as the Jacobian matrix of a map F. Let $C_1 \pm C_2 = \{z_1 \pm z_2 \,|\, z_1 \in C_1, z_2 \in C_2\}$ be the Minkowski sum/minus of sets C_1 and C_2, and rint(C) be the relative interior of a convex set C (Rockafellar & Wets, 1998, page 25 and page 64).

2. Preliminaries

In this section, we give some preliminary knowledge related to convex analysis, variational inequality, and graph theory.

A set $C \subseteq \mathbb{R}^n$ is *convex* if $\lambda z_1 + (1-\lambda)z_2 \in C$ for any $z_1, z_2 \in C$ and $0 \le \lambda \le 1$. For a closed convex set C, the *projection* map $P_C : \mathbb{R}^n \to C$ is defined as

$$P_C(x) \triangleq \operatorname*{argmin}_{y\in C} \|x-y\|.$$

The following two basic properties hold:

$$(x - P_C(x))^T(P_C(x) - y) \ge 0, \quad \forall y \in C, \tag{1}$$

$$\|P_C(x) - P_C(y)\| \le \|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{2}$$

For $x \in C$, the *tangent cone* to C at x is

$$\mathcal{T}_C(x) \triangleq \left\{ \lim_{k\to\infty} \frac{x_k - x}{t_k} \,\middle|\, x_k \in C,\ t_k > 0,\ \text{and } x_k \to x,\ t_k \to 0 \right\}.$$

and the *normal cone* to C at x is

$$\mathcal{N}_C(x) \triangleq \{v \in \mathbb{R}^n \,|\, v^T(y - x) \le 0,\ \text{for all } y \in C\}.$$

Lemma 1 (Rockafellar & Wets 1998, Theorem 6.42). *Let C_1 and C_2 be two closed convex subsets of $\mathbb{R}^n$. If $0 \in$ rint$(C_1 - C_2)$, then*

$$\mathcal{T}_{C_1\cap C_2}(x) = \mathcal{T}_{C_1}(x) \cap \mathcal{T}_{C_2}(x),\ \forall x \in C_1 \cap C_2.$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if $f(\lambda z_1 + (1-\lambda)z_2) \le \lambda f(z_1) + (1-\lambda)f(z_2)$ for any $z_1, z_2 \in C$ and $0 \le \lambda \le 1$. A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *monotone (strictly monotone)* on a set Ω if $(x - y)^T(F(x) - F(y)) \ge 0\, (> 0)$ for all $x, y \in \Omega$ and $x \ne y$. A differentiable map F is monotone if and only if the Jacobian matrix $\mathcal{J}F(x)$ (not necessarily symmetric) is positive semidefinite for each x (Rockafellar & Wets, 1998, Theorem 12.3).

Given a subset $\Omega \subseteq \mathbb{R}^n$ and a map $F : \Omega \to \mathbb{R}^n$, the *variational inequality*, denoted by VI(Ω, F), is to find a vector $x \in \Omega$ such that

$$(y - x)^T F(x) \ge 0, \quad \forall y \in \Omega,$$

and the set of solutions to this problem is denoted by SOL(Ω, F) (Facchinei & Pang, 2003). When Ω is closed and convex, the solution of VI(Ω, F) can be equivalently reformulated via projection as follows:

$$x \in \mathrm{SOL}(\Omega, F) \Leftrightarrow x = P_\Omega(x - F(x)). \tag{3}$$

Lemma 2 (Facchinei & Pang, 2003, Corollary 2.2.5, and Theorem 2.2.3). *Consider VI(Ω, F), where the set $\Omega \subset \mathbb{R}^n$ is convex and the map $F : \Omega \to \mathbb{R}^n$ is continuous. The following two statements hold:*

(1) *if Ω is compact, then SOL(Ω, F) is nonempty and compact;*
(2) *if Ω is closed and $F(x)$ is strictly monotone, then VI(Ω, F) has at most one solution.*

The following lemma about a regularized gap function is important for our results.

Lemma 3 (Fukushima, 1992). *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map and $H(x) = P_\Omega(x - F(x))$. Define $g : \mathbb{R}^n \to \mathbb{R}$ as*

$$g(x) = (x - H(x))^T F(x) - \frac{1}{2}\|x - H(x)\|^2.$$

Then $g(x) \ge 0$ is differentiable and its gradient is

$$\nabla g(x) = F(x) + (\mathcal{J}F(x) - I_n)(x - H(x)).$$

Furthermore, it is known that the information exchange among agents can be described by a graph. A graph with node set $\mathcal{V}$ and edge set $\mathcal{E}$ is written as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (Godsil & Royle, 2001). If agent $i \in \mathcal{V}$ can receive information from agent $j \in \mathcal{V}$, then $(j, i) \in \mathcal{E}$ and agent j belongs to agent i's neighbor set $\mathcal{N}_i = \{j \,|\, (j, i) \in \mathcal{E}\}$. $\mathcal{G}$ is said to be *undirected* if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$, and $\mathcal{G}$ is said to be *connected* if any two nodes in $\mathcal{V}$ are connected by a path (a sequence of distinct nodes in which any consecutive pair of nodes share an edge).

3. Problem formulation

Consider an N-player aggregative game with coupled constraints as follows. For $i \in \mathcal{V} \triangleq \{1, \ldots, N\}$, the ith player aims to minimize its cost function $J_i(x_i, x_{-i}) : \Omega \to \mathbb{R}$ by choosing the local decision variable x_i from a local strategy set $\Omega_i \subset \mathbb{R}^{n_i}$, where $x_{-i} \triangleq col(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$, $\Omega \triangleq \Omega_1 \times \cdots \times \Omega_N \subset \mathbb{R}^n$ and $n = \sum_{i\in\mathcal{V}} n_i$. The *strategy profile* of this game is $x \triangleq col(x_1, \ldots, x_N) \in \Omega$. The *aggregation* map $\sigma : \mathbb{R}^n \to \mathbb{R}^m$, to specify the cost function as $J_i(x_i, x_{-i}) = \vartheta_i(x_i, \sigma(x))$ with a function $\vartheta_i : \mathbb{R}^{n_i+m} \to \mathbb{R}$, is defined as

$$\sigma(x) \triangleq \frac{1}{N}\sum_{i=1}^{N} \varphi_i(x_i), \tag{4}$$