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Brief paper

On the triangular canonical form for uniformly observable controlled systems☆

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ABSTRACT

We study controlled systems which are uniformly observable and differentially observable with an order larger than the system state dimension. We establish that they may be transformed into a (partial) triangular canonical form but with possibly non locally Lipschitz functions. We characterize the points where this Lipschitzness may be lost and investigate the link with uniform infinitesimal observability.

1. Introduction

1.1. Context

A lot of attention has been dedicated to the construction of nonlinear observers. Although a general theory has been obtained for linear systems, very few general approaches exist for nonlinear systems. In particular, the theory of high gain (Khalil & Praly, 2013 and references therein) and Luenberger (Andrieu, 2014; Andrieu & Praly, 2006) observers have been developed for autonomous nonlinear systems but their extension to controlled systems is not straightforward.

For designing an observer for a system, a preliminary step is often required. It consists in finding a reversible coordinate transformation, allowing us to rewrite the system dynamics in a targeted form more favorable for writing and/or analyzing the observer. In presence of input, two tracks are possible depending on whether we consider the input as a simple time function, making the system time dependent or as a more involved infinite dimensional parameter, making the system a family of dynamical systems, indexed by the input. In the former case, the transformation mentioned above is considered time dependent, and thus may need to be redesigned for each input. In the latter case, the transformation can be input-dependent. Specifically :

- in Jouan and Gauthier (1996) (see also Gauthier & Kupka, 2001), the transformation depends on the inputs and its derivatives. When the *ACP(N) condition* is verified (see Lemma (Gauthier & Kupka, 2001, Definition 5.2.1, Lemma 5.2.2)), it leads to the so called phase-variable representation as targeted form (see Gauthier & Kupka, 2001, Definition 2.3.1), for which a high gain observer can be built.
- in Besançon (1999), the transformation does not depend on the input, and leads to a block triangular form when the system verifies the observability rank condition (see Hermann and Krener, 1977). However, afterwards, an extra condition on the input is needed for the observer design.
- in Gauthier and Bornard (1981) and Gauthier, Hammouri, and Othman (1992), the transformation does not depend on the input, and leads to a triangular form when the system is (a) uniformly observable (see Gauthier & Kupka, 2001, Definition I.2.1.2 or Definition 2), and (b) strongly differentially observable of order equal to the system state dimension (see Definition 1). This so-called observable canonical form allows the design of a high gain observer.

In this paper, we complete and detail the results announced in Bernard, Praly, and Andrieu (2016). We work within the third context (of the second track), but going beyond (Gauthier & Kupka, 2001) with allowing the strong differential observability order to

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be larger than the system state dimension. We shall see that, in this case, the system dynamics may still be described by a (partial) triangular canonical form (see (3)) but with functions which may be non locally Lipschitz.

1.2. Definitions and problem statement

We consider a controlled system of the form :

$$\dot{x} = f(x) + g(x)u \quad , \qquad y = h(x) \tag{1}$$

where x is the state in $\mathbb{R}^{d_x}$, u is an input in $\mathbb{R}^{d_u}$, y is a measured output in $\mathbb{R}$ and the functions f, g and h are sufficiently many times differentiable, f being a column d_x-dimensional vector field and g a $(d_x \times d_u)$-dimensional matrix field. In the following, for a scalar function α, $L_f\alpha$ denotes its Lie derivative in the direction of f. It has scalar values. We denote

$$\mathbf{H}_i(x) = (h(x), L_f h(x), \ldots, L_f^{i-1}h(x)) \in \mathbb{R}^i . \tag{2}$$

It is a column i-dimensional vector. Similarly $L_g\alpha$ denotes the Lie derivative along each of the d_u columns of g. It has row d_u-dimensional vector values.

Given an input time function $t \mapsto u(t)$ taking values in a compact subset $\mathcal{U}$ of $\mathbb{R}^{d_u}$, we denote $X_u(x, t)$ a solution of (1) going through x at time 0. We are interested in solving :

Problem $\mathcal{P}$: *Given a compact subset $\mathcal{C}$ of $\mathbb{R}^{d_x}$, under which condition do there exist integers $\mathfrak{T}$ and d_z, a continuous injective function $\Psi : \mathcal{C} \to \mathbb{R}^{d_z}$, and continuous functions $\varphi_{d_z} : \mathbb{R}^{d_z} \to \mathbb{R}$ and $\mathfrak{g}_i : \mathbb{R}^i(\text{or } \mathbb{R}^{d_z}) \to \mathbb{R}^{d_u}$ such that, when x is in $\mathcal{C}$ and satisfies (1) and u is in $\mathcal{U}$, $z = \Psi(x)$ satisfies*

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 + \mathfrak{g}_1(z_1)u \\ &\vdots& \\ \dot{z}_{\mathfrak{T}} &=& z_{\mathfrak{T}+1} + \mathfrak{g}_{\mathfrak{T}}(z_1, \ldots, z_{\mathfrak{T}})u \\ \dot{z}_{\mathfrak{T}+1} &=& z_{\mathfrak{T}+2} + \mathfrak{g}_{\mathfrak{T}+1}(z)u \\ &\vdots& \\ \dot{z}_{d_z} &=& \varphi_{d_z}(z) + \mathfrak{g}_{d_z}(z)u \end{array} \quad , \quad y = x_1 \tag{3}$$

Because $\mathfrak{g}_i$ depends only on z_1 to z_i, for $i \leq \mathfrak{T}$, but potentially on all the components of z for $i > \mathfrak{T}$, we call this particular form up-to-$\mathfrak{T}$-triangular canonical form and $\mathfrak{T}$ is called the order of triangularity. When $d_z = \mathfrak{T} + 1$, we say full triangular canonical form. When the functions φ_{d_z} and $\mathfrak{g}_j$ are locally Lipschitz we say Lipschitz up-to-$\mathfrak{T}$-triangular canonical form.

We are interested in addressing the Problem $\mathcal{P}$ because, when the functions are Lipschitz and $d_z = \mathfrak{T} + 1$, we get the nominal form for which high gain observers can be designed and therefore $X_u(x, t)$ can be estimated knowing y and u as long as $(X_u(x, t), u(t))$ is in the given compact set $\mathcal{C} \times \mathcal{U}$.

We will use the following two notions of observability:

Definition 1 (*Differential Observability*[1]). System (1) is weakly differentially observable of order $\mathcal{O}$ on an open subset $\mathcal{S}$ of $\mathbb{R}^{d_x}$ if the function $\mathbf{H}_{\mathcal{O}}$ (see (2)) is injective on $\mathcal{S}$. If it is also an immersion, the system is called strongly differentially observable of order $\mathcal{O}$.

Definition 2 (*Uniform Observability*). (See Gauthier & Kupka, 2001, Definition I.2.1.2.) System (1) is uniformly observable on an open subset $\mathcal{S}$ of $\mathbb{R}^{d_x}$ if, for any pair (x_a, x_b) in $\mathcal{S}^2$ with $x_a \neq x_b$, any strictly positive number T, and any C^1 function u defined on $[0, T)$, there exists a time $t < T$ such that $h(X_u(x_a, t)) \neq h(X_u(x_b, t))$ and $(X_u(x_a, s), X_u(x_b, s)) \in \mathcal{S}^2$ for all $s \leq t$.

[1] This notion is weaker than the usual differential observability defined for instance in Gauthier and Kupka (2001, Definition I.2.4.2) for controlled systems, because it is a differential observability of the drift system only, namely when $u \equiv 0$.

Note that this notion is a matter of instantaneous observability since T can be arbitrarily small. In the case where $\mathbf{H}_{d_x}$ is a diffeomorphism, we have

Proposition 1 (*See Gauthier & Bornard, 1981; Gauthier et al., 1992*). *If System (1) is uniformly observable and strongly differentially observable of order $\mathcal{O} = d_x$ on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$, it can be transformed on $\mathcal{C}$ into a full Lipschitz triangular canonical form of dimension $d_z = d_x$.*

In general, it is possible for the system not to be strongly differentially observable of order d_x everywhere. This motivates our interest in the case where the system is strongly differentially observable of order $\mathcal{O} > d_x$, i.e. $\mathbf{H}_{\mathcal{O}}$ is an injective immersion and not a diffeomorphism. As we shall see in Section 2, in this case, we may still have an at least up-to-$(d_x + 1)$-triangular form, but the Lipschitzness of its nonlinearities can be lost. Since this property is crucial for the implementation of high gain observers (see Ciccarella, Mora, & Germani, 1993), we give in Section 3 some sufficient conditions under which the Lipschitzness is preserved.

2. Immersion case ($\mathcal{O} > d_x$)

The specificity of the triangular canonical form (3) is not so much in its structure but more in the dependence of its functions $\mathfrak{g}_i$ and φ_{d_z}. Indeed, by choosing $\Psi = \mathbf{H}_{d_z}$, we obtain:

$$\dot{\overline{\mathbf{H}_{d_z}(x)}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \mathbf{H}_{d_z}(x) + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ L_f^{d_z}h(x) \end{pmatrix} + L_g\mathbf{H}_{d_z}(x)u$$

To get (3), we need further the existence of a function φ_{d_z} satisfying

$$L_f^{d_z}h(x) = \varphi_{d_z}(\mathbf{H}_{d_z}(x)) \quad \forall x \in \mathcal{C} \tag{4}$$

and, for $i \leq \mathfrak{T}$, of functions $\mathfrak{g}_i$ satisfying

$$L_g L_f^{i-1}h(x) = \mathfrak{g}_i(h(x), \ldots, L_f^{i-1}h(x)) \quad \forall x \in \mathcal{C}. \tag{5}$$

Let us illustrate via the following elementary example what can occur.

Example 1. Consider the system defined as

$$\dot{x}_1 = x_2 , \quad \dot{x}_2 = x_3^3 , \quad \dot{x}_3 = 1 + u , \quad y = x_1.$$

We get

$$\mathbf{H}_3(x) = (h(x), L_f h(x), L_f^2 h(x)) = (x_1, x_2, x_3^3)$$
$$\mathbf{H}_5(x) = (\mathbf{H}_3(x), L_f^3 h(x), L_f^4 h(x)) = (\mathbf{H}_3(x), 3x_3^2, 6x_3).$$

Hence $\mathbf{H}_3$ is a bijection and $\mathbf{H}_5$ is an injective immersion on $\mathbb{R}^3$. So this system is weakly differentially observable of order 3 on $\mathbb{R}^3$ and strongly differentially observable of order 5 on $\mathbb{R}^3$. Also the function $(x_1, x_2, x_3) \mapsto (y, \dot{y}, \ddot{y})$ being injective for all u, it is uniformly observable on $\mathbb{R}^3$. From this we could be tempted to pick $d_z = 3$ or 5 and the compact set $\mathcal{C}$ arbitrary in $\mathbb{R}^3$. Unfortunately, if we choose $d_z = 3$, we have

$$L_f^3 h(x) = 3x_3^2 = 3(L_f^2 h(x))^{2/3}$$

and there is no locally Lipschitz function φ_3 satisfying (4) if the given compact set $\mathcal{C}$ contains a point satisfying $x_3 = 0$. If we choose $d_z = 5$, we have

$$L_g L_f^2 h(x) = 3x_3^2 = L_f^3 h(x) = 3(L_f^2 h(x))^{2/3}$$

and there is no locally Lipschitz function $\mathfrak{g}_3$ satisfying (5) if the given compact set $\mathcal{C}$ contains a point satisfying $x_3 = 0$.