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Brief paper

Partial state reachability of multiple linear systems in series ☆

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ABSTRACT

We explore the issue of characterizing reachability for a subset of the component systems in the series connection of multivariable linear discrete-time systems. Using tools from algebraic systems theory, partial state reachability of a series connection is characterized in terms of Toeplitz operators and coprime factorizations of the component transfer functions. Our results extend earlier results by Callier and Nahum (1975) on reachability of the series connection of two systems, as well as a more recent characterization in Fuhrmann and Helmke (2013) of the full reachability on the series connection of $r \geq 2$ linear systems.

1. Introduction

In controlling systems in a network, typically one may only be interested in controlling specific subsystems. Such situations occur when the subsystems of interest are linked together and the links between the nodes of interest exhibit their own dynamics, modeling communication, transport or computation delays (e.g., for coding/decoding) (Ji, Wang, Lin, & Wang, 2010), for instance in multi-agent robotic systems (Ji & Egerstedt, 2006), and control of passively interconnected chains (Yamamoto & Smith, 2016). Reachability of these link dynamical states is not of primary interest. More generally, controlling precisely all the states of a large network is rarely a goal in itself, as is the case in the Supervisory Control and Data Acquisition (SCADA) system (Liu, Xiao, Li, Liang, & Chen, 2012). While a notion of partial reachability seems like a more relaxed condition of reachability, its precise formulation poses fascinating challenges. In this paper we restrict the discussion to the series connection of several systems.

A more abstract formulation of this problem is the control of an arbitrary subset of component systems in the series connection of a series of systems, Σ_1 to Σ_r. Assume that the ith component has a state space realization

$$\Sigma_i: \begin{aligned} x_i(t+1) &= A_i x_i + B_i u_i(t) \\ w_i(t) &= C_i x_i(t). \end{aligned} \tag{1}$$

with *minimal* realization $(A_i, B_i, C_i) \in \mathbb{F}^{n_i\times n_i} \times \mathbb{F}^{n_i\times m_i} \times \mathbb{F}^{p_i\times n_i}$, $i = 1, 2, \ldots, r, m := m_1$. Set $\overline{n} = \sum_{i=1}^r n_i$. The coupling structure of the series connection requires the identifications of the subsystems' external signals $u(t) = u_1(t), \quad w_i(t) = u_{i+1}(t), \quad i = 1, \ldots, r$, thus leading to the overall system equations

$$\begin{aligned} x(t+1) &= \mathcal{A}x(t) + \mathcal{B}u(t) \\ y(t) &= \mathcal{C}x(t) \end{aligned} \tag{2}$$

with system matrices

$$\mathcal{A} = \begin{bmatrix} A_1 & & & & \\ B_2C_1 & A_2 & & & \\ & B_3C_2 & A_3 & & \\ & & \ddots & \ddots & \\ & & & B_rC_{r-1} & A_r \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$
$$\mathcal{C} = [0, \quad \cdots\cdots \quad 0, \quad C_r\ ].$$

The contribution of this paper is a *precise characterization of the reachability of a subset of systems* in a series connection. Let $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$ be a subset of the index set $\{1, \ldots, r\}$, ordered as $1 \leq i_1 < \cdots < i_k \leq r$. We refer to the reachability of the states of the subsystems indexed by $\mathcal{I}$ as partial state reachability of the series connection $\Sigma_1 \to \cdots \to \Sigma_r$.

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Definition 1. The $\Sigma_1 \to \cdots \to \Sigma_r$ series connection is $\mathcal{I}$-partial state reachable if for arbitrary $x_{i_1} \in \mathbb{F}^{n_{i_1}}, \ldots, x_{i_k} \in \mathbb{F}^{n_{i_k}}$, there exists an integer $T \geq 0$ and an input sequence $u(0), u(1), \ldots, u(T-1)$ so that for all $i_j \in \mathcal{I}$, the final state (at step T) of the subsystem Σ_{i_j} is x_{i_j}, given that the entire series connection is initially in the zero state.

By the minimality assumption, each of the systems Σ_i with $i \in \mathcal{I}$ is reachable, which is clearly a necessary condition for $\mathcal{I}$-partial state reachability. Full reachability of the series connection of two systems was first characterized by Callier and Nahum (1975), and has been subsequently extended by Fuhrmann and Helmke (2013, 2015) for an arbitrary number of systems in a series connection. Partial state reachability is a special case of output reachability, where the read-out matrix C projects the full state state space onto the state space of $\Sigma_{i_1}, \Sigma_{i_2}, \ldots, \Sigma_{i_k}$. The classical characterization of output reachability may be found in Sontag (1998). However, its implementation as a rank test does not take the coupling structure of series connections into account and therefore does not help in solving the problem considered here. The problem of characterizing $(1, 3)$-partial state reachability in a series connection of three systems was first addressed in Verriest, Helmke, and Fuhrmann (2016), and it was shown not to be trivial.
A state space characterization for partial reachability is readily obtained. For a realization (A_i, B_i, C_i), define the associated $\infty \times \infty -$ Toeplitz matrix as

$$\mathbb{T}_i = \mathbb{T}(A_i, B_i, C_i) = \begin{bmatrix} 0 & H_1^{(i)} & H_2^{(i)} & \cdots \\ 0 & 0 & H_1^{(i)} & \ddots \\ \vdots & & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix},$$

where $H_j^{(i)} := C_i A_i^{j-1} B_i$ denotes the jth Markov parameter of subsystem Σ_i. Let $R(A_i, B_i)$ denote the infinite length $n_i \times \infty$–reachability matrix. The series connection $\Sigma_1 \to \Sigma_2 \to \Sigma_3 \to \cdots \to \Sigma_N$ is $\mathcal{I}$-partial state reachable if and only if the operator

$$R_{\mathcal{I}} = \begin{bmatrix} R(A_{i_1}, B_{i_1})\mathbb{T}_{i_1-1} \cdots \mathbb{T}_1 \\ R(A_{i_2}, B_{i_2})\mathbb{T}_{i_2-1} \cdots \mathbb{T}_1 \\ \vdots \\ R(A_{i_k}, B_{i_k})\mathbb{T}_{i_k-1} \cdots \mathbb{T}_1 \end{bmatrix} \tag{4}$$

has full row rank (if $i_1 = 1$, then the first row has to be replaced by $R(A_1, B_1)$). The above result follows by a straightforward computation, using the fact that the Toeplitz matrix of the (proper) series connection $\Sigma_1 \to \Sigma_2$ is the product of the individual Toeplitz matrices $\mathbb{T}(A_i, B_i, C_i)$ (in correct order). However this result is somewhat premature as it is not particularly useful. The rank condition is hard to check as $R_{\mathcal{I}}$ essentially is an operator, with the *number of columns not a priori determined*. Our main result, Theorem 3 below, yields an efficient formulation using polynomial matrix representations and tools from algebraic system theory. For space reasons this cannot be reviewed here, and we direct the reader to Rosenbrock (1970) and Fuhrmann and Helmke (2015) for the necessary background on the Rosenbrock representation, its shift realization, the polynomial model, the reachability map, and Fuhrmann system equivalence.

The paper is organized as follows. Section 2 gives a precise characterization of the notion of partial state reachability. In Section 3, we present our main results on partial state reachability: the Toeplitz operator characterization and equivalent characterizations in polynomial matrix fashion. Examples illustrate the concepts and a complete solution to the (2)- and (1, 3)- partial reachability problem is sketched.

2. Polynomial description of series connections

Let $\mathbb{F}((z^{-1}))^m$ denote the vector space of truncated Laurent series, i.e., $f(z) = \sum_{j=-\infty}^{n_f} f_j z^j, f_j \in \mathbb{F}^m$. Thus f_{-1} denotes the residue of $f(z)$. We denote the canonical projections onto the strictly proper and polynomial parts, respectively, by $\pi_- : \mathbb{F}((z^{-1}))^m \longrightarrow z^{-1}\mathbb{F}[[z^{-1}]]^m$ and $\pi_+ : \mathbb{F}((z^{-1}))^m \longrightarrow \mathbb{F}[z]^m$. For a nonsingular polynomial matrix $T(z) \in \mathbb{F}[z]^{m\times m}$, define a linear projection map $\pi_T : \mathbb{F}[z]^m \longrightarrow \mathbb{F}[z]^m$ by

$$\pi_T f = T\pi_-(T^{-1}f), \qquad f \in \mathbb{F}[z]^m.$$

The space $X_T := \operatorname{Im} \pi_T$ is called the **polynomial model** of $T(z)$. It is a finite dimensional vector space. Observe that in the case of $T(z) = (zI - A)$, the image of $B\mathbb{F}[z]^m$ under the projection $\pi_{(zI-A)}$ is precisely the (finite-dimensional) subspace of states associated with the first-order model $x_{k+1} = Ax_k + Bu_k$ that can be reached from rest by the action of inputs with finite support. Let $\mathbb{T}_G : \mathbb{F}[z]^m \to \mathbb{F}[z]^p$ denote the Toeplitz operator $\mathbb{T}_G u := \pi_+(Gu)$ with symbol $G(z)$.

Consider now r decoupled minimal discrete–time systems (1), and let $G_i(z) = C_i(zI - A_i)^{-1}B_i$ denote their associated transfer matrix. The series connection of these systems then has the state space representation $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as in (3), with transfer matrix

$$G(z) = G_r(z) \cdots G_1(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}.$$

In terms of right coprime factorizations $G_i(z) = N_i(z)D_i(z)^{-1}, i = 1, \ldots, r$, one has a polynomial matrix fraction description of the series connection as $G(z) = V(z)T(z)^{-1}U(z) + W(z)$, where

$$T(z) = \begin{bmatrix} D_1(z) & 0 & \cdots & 0 \\ -N_1(z) & D_2(z) & & \\ & \ddots & \ddots & \\ 0 & \cdots & -N_{r-1}(z) & D_r(z) \end{bmatrix}, \quad U(z) = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

and

$$V(z) = \begin{bmatrix} 0 & 0 & \cdots & 0 & N_r(z) \end{bmatrix}, \qquad W(z) = 0.$$

It is easily seen by inspection, see Fuhrmann (1977) and Fuhrmann and Helmke (2015) for the terminology and further details, that the two polynomial system matrices

$$\left[\begin{array}{c|c} zI - \mathcal{A} & -\mathcal{B} \\ \hline \mathcal{C} & 0 \end{array}\right], \quad \left[\begin{array}{c|c} T(z) & -U(z) \\ \hline V(z) & 0 \end{array}\right]$$

are Fuhrmann strict system equivalent and thus their shift realizations are similar. This implies that the shift realization (A, B, C) of (U, T, V, W) is state space equivalent to $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Following Fuhrmann and Helmke (2015), an isomorphism between the respective state spaces is given as $Z : X_T \longrightarrow X_{zI-\mathcal{A}}, \quad Zf = \pi_{zI-\mathcal{A}}(\overline{B}f)$, where $\overline{B} := \operatorname{diag}(B_1, \ldots, B_r)$. A straightforward computation reveals that

$$Zf = \begin{bmatrix} \pi_{zI-A_1}(B_1 f_1) \\ \pi_{zI-A_2}(B_2(f_2 + \pi_+(G_1 f_1))) \\ \vdots \\ \pi_{zI-A_r}(B_r(f_r + \pi_+(G_{r-1}f_{r-1}) + \cdots \\ + \cdots + \pi_+(G_{r-1}\cdots G_1 f_1))) \end{bmatrix}.$$

Moreover, $f \in X_T$ if and only if for all $i = 1, \ldots, r$

$$f_i + \pi_+(G_{i-1}f_{i-1}) + \cdots + \pi_+(G_{i-1}\cdots G_1 f_1) \in X_{D_i}.$$

The map Z intertwines the shift realizations (T, U, V) and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and therefore maps the reachability subspace $T_1 X_{T_2}$ of (A, B, C) isomorphically onto that of $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Here $T_1 = gcld(T, U)$ and $T_1T_2 = T$. In particular, the respective reachability maps satisfy for all $u \in \mathbb{F}[z]^m$: $Z(\pi_T(Uu)) = \pi_{zI-\mathcal{A}}(\mathcal{B}u)$. Let $\overline{n}_{\mathcal{I}} = n_{i_1} + \cdots + n_{i_k}$ and let $\mathrm{pr}_{\mathcal{I}} : \mathbb{F}^{\overline{n}} \to \mathbb{F}^{\overline{n}_{\mathcal{I}}}$ denote the canonical projection onto