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Brief paper

Stability of homogeneous nonlinear systems with sampled-data inputs☆

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ABSTRACT

The main goal of this article is to use properties of homogeneous systems for addressing the problem of stability for a class of nonlinear systems with sampled-data inputs. This nonlinear strategy leads to several kinds of stability, i.e. local asymptotic stability, global asymptotic stability or global asymptotic set stability, depending on the sign of the degree of homogeneity. The results are illustrated with the case of the double integrator.

1. Introduction

Feedback control systems wherein the control loops are closed through a real-time network are called Networked Control Systems (NCSs) (Zhang, Branicky, & Phillips, 2001). In NCSs, the presence of packet-based communication, network delays, limited bandwidth and packet dropouts is inevitable. NCSs are widely studied in automatic control since several years (Bemporad, Heemels, & Johansson, 2010; Wang & Liu, 2008).

When studying NCSs, a crucial problem is to determine whether some stability properties pertain through sampled-data inputs (Walsh, Ye, & Bushnell, 2002). Moreover, due to, for instance, data packet dropouts (Liu, Fridman, & Hetel, 2014), the inputs are not only sampled but also usually nonuniformly sampled with respect to time and this is called aperiodic sampled-data inputs (Hetel, Fiter, Omran, Seuret, Fridman, Richard, & et al., 2017). Several approaches have been developed to solve the problem of stability for systems with sampled-data inputs, such as the input/output approach (Nešić & Teel, 2004b; Omran, Hetel, Richard, & Lamnabhi-Lagarrigue, 2013) or the sum of squares approach (Bauer, Maas, & Heemels, 2012). Moreover, different dynamical models as hybrid systems (Donkers, Heemels, Van De Wouw, & Hetel, 2011; Zhang et al., 2001), discrete-time systems (Nešić & Teel, 2004a) or time-delay systems (Fujioka, 2009; Liu et al., 2014; Mazenc, Malisoff, & Dinh, 2013; Seuret, 2012) have also been used to tackle this problem. Most of these strategies (Bauer et al., 2012; Nešić & Teel, 2004b; Omran et al., 2013) are developed under the emulation method where the controller is first designed in continuous time and then implemented as a sampled-data controller (Bemporad et al., 2010; Nešić, Teel, & Carnevale, 2009).

The notion of homogeneity was introduced in Rothschild and Stein (1976) and Zubov (1958) and developed by many authors, for instance in Bhat and Bernstein (2005), Kawski (1995) and Moulay (2009). In Anta and Tabuada (2010), a new nonlinear approach based on the input-to-state stability (ISS) properties of homogeneous systems has been introduced for studying self-triggered control for nonlinear systems. We know from Laila, Nešić, and Teel (2002) and Nešić, Teel, and Sontag (1999) that the emulation of a feedback law, which globally asymptotically stabilizes the origin of a nonlinear continuous-time system, leads in general to semi-global practical stability in the sampled-data context. In our article, we show that if the nonlinear system satisfies in addition a homogeneity property, then the global asymptotic stability, local asymptotic stability or global asymptotic set stability is preserved under emulation depending on the degree of homogeneity.

The main result shows that if it is possible to build a stabilizing static feedback control for a continuous system such that the

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closed-loop system with sampled-data inputs satisfies a homogeneity property, then it is possible to preserve different notions of stability that depend on the degree of homogeneity κ for the closed-loop system with aperiodic sampled-data inputs. When $\kappa = 0$, the closed-loop system is globally asymptotically stable if the maximum sampling period h is shorter than a constant H. Linear systems are special cases of homogeneous systems of degree 0 and our result hence shows that homogeneity explains this feature of linear systems. We refer to Seuret (2012), Suh (2008) and Zhang et al. (2001) for different strategies used in the literature for estimating H in the linear case. When $\kappa > 0$, the closed-loop system achieves local asymptotic stability, while when $\kappa < 0$, it achieves global asymptotic set stability. In these two cases, the results remain true regardless of $h < +\infty$, although the size of the domain of attraction and of the limit set do depend on h. With the use of a homogeneous observer, the results are then extended to continuous systems with an output. The results are finally applied to the case of the double integrator which is rather important in control theory despite its simplicity (Bernuau, Perruquetti, Efimov, & Moulay, 2015; Bhat, Bernstein, et al., 1998).

The article is organized as follows. After some notations and definitions given in Section 2, we develop the main results of the article in Section 3. The example of the double integrator is treated in Section 4. Finally, a conclusion is addressed in Section 5.

2. Notations and definitions

Let us introduce the following notations:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.
- For $r_1, r_2, \ldots, r_n$, $\mathrm{Diag}(r_1, \ldots, r_n)$ denotes the diagonal matrix of dimension $n \times n$ with k th diagonal entry r_k.
- A measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ is *locally essentially bounded* if for any $0 \leq a < b$, the function d is essentially bounded on the segment $[a, b]$. $\mathcal{L}^\infty_{\mathrm{loc}}$ denotes the set of locally essentially bounded functions $d : \mathbb{R}_+ \to \mathbb{R}^m$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded.
- A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and if for each fixed $s \in \mathbb{R}_+$ the function $t \mapsto \beta(s, t)$ is decreasing to 0.
- The notation $d_x V$ (resp. $d_x \Phi$) stands for the differential of the function V (resp. the diffeomorphism Φ) at the point x.
- $\lfloor x \rceil^\alpha = |x|^\alpha \,\mathrm{sign}(x)$ where $x \in \mathbb{R}$ and $\alpha > 0$.
- Given that we will deal with a variety of suprema, we will use a compact notation in the computations. For instance, the notation $\sup\{f(x, y) : g(x) = 0, h(y) \leq 1\}$ stands for $\sup_{x \in E, y \in F} f(x, y)$ where $E = \{x \in \mathbb{R}^n : g(x) = 0\}$ and $F = \{y \in \mathbb{R}^n : h(y) \leq 1\}$.

Consider the following system with f continuous

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{1}$$

Let us recall the definitions of Lyapunov set stability given for instance in Bhatia and Szegö (2002).

Definition 1. A compact set $K \subset \mathbb{R}^n$ is:

- *stable* w.r.t. the system (1) if for any neighborhood U of K, there exists a neighborhood W of K such that for any maximal solution $x(t)$ of (1), if there exists t_0 such that $x(t_0) \in W$, then $x(t)$ is defined for all $t \geq t_0$ and $x(t) \in U$ for all $t \geq t_0$;
- *locally attractive* w.r.t. the system (1) if there exists a neighborhood U of K such that for any maximal solution $x(t)$ of (1), if there exists t_0 such that $x(t_0) \in U$, then $x(t)$ is defined for all $t \geq t_0$ and $x(t) \to K$ when $t \to +\infty$ (the *domain of attraction* of a locally attractive set is the biggest set U for which the preceding point holds);
- *globally attractive* w.r.t. the system (1) if it is locally attractive and if its domain of attraction is $\mathbb{R}^n$;
- *locally* (resp. *globally*) *asymptotically stable* w.r.t. the system (1) if it is stable and locally (resp. globally) attractive w.r.t. the system (1);
- *unstable* if it is not stable.

Let us consider the following nonlinear system

$$\dot{x} = f(x, \Delta) \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $\Delta \in \mathcal{L}^\infty_{\mathrm{loc}}$ is the external input and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. Let us recall the definition of input-to-state stability given for instance in Sontag and Wang (1995).

Definition 2. The system (2) is called *input-to-state stable (ISS)* if there exist some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any input $\Delta \in \mathcal{L}^\infty_{\mathrm{loc}}$ and any $x_0 \in \mathbb{R}^n$ it holds that

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \operatorname*{ess\,sup}_{\tau \in [0,t]} \gamma(\|\Delta(\tau)\|) \quad \forall t \geq 0$$

where $x(t)$ is the solution of the system (2) satisfying $x(0) = x_0$. The function γ is called a *nonlinear asymptotic gain*.

The most common notion of homogeneity is the *weighted homogeneity*, based on a particular choice of the coordinates, while the most generic one is the *geometric homogeneity*, which is coordinate free (Bhat & Bernstein, 2005; Kawski, 1995). We use in the sequel the framework of geometric homogeneity.

Definition 3. A vector field ν on $\mathbb{R}^n$ is called an *Euler vector field* if ν is of class C^1, complete (i.e. the maximal solutions of $\dot{x} = \nu(x)$ are defined on $\mathbb{R}$) and if the origin is a globally asymptotically equilibrium of $-\nu$.

Definition 4. Let ν be an Euler vector field on $\mathbb{R}^n$, $\Phi^s(x)$ denotes the value of the flow[1] of ν at time s with initial condition x. A function $V : \mathbb{R}^n \to \mathbb{R}$ is ν-homogeneous of degree $\kappa \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ and all $s \in \mathbb{R}$ we have $V(\Phi^s(x)) = e^{\kappa s} V(x)$. A vector field f on $\mathbb{R}^n$ is ν-homogeneous of degree $\kappa \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ and all $s \in \mathbb{R}$ we have $f(\Phi^s(x)) = e^{\kappa s} d_x \Phi^s f(x)$.

A direct application of the chain rule then gives us the following result.

Proposition 5. *Let ν be an Euler vector field on $\mathbb{R}^n$. If $V : \mathbb{R}^n \to \mathbb{R}$ is differentiable and ν-homogeneous of degree κ, then for all $x \in \mathbb{R}^n$ and all $s \in \mathbb{R}$*

$$d_{\Phi^s(x)} V \cdot d_x \Phi^s = e^{\kappa s} d_x V.$$

Remark 6. If we consider a matrix $A \in \mathbb{R}^{n \times n}$ such that $-A$ is Hurwitz then the vector field defined by $\nu(x) = Ax$ is an Euler vector field and the flow of ν verifies $\Phi^s(x) = \exp(As)x$. In particular, if $A = \mathrm{Diag}(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n > 0$, the vector field $\nu(x) = Ax$ is Euler and we find $\Phi^s(x) = \mathrm{Diag}(e^{r_1 s}, \ldots, e^{r_n s})x$. The homogeneity defined by such an Euler vector field is usually referred to as *weighted homogeneity*, the coefficients $r_1, \ldots, r_n$ are called the weights and $\mathbf{r} = [r_1, \ldots, r_n]$ is called the generalized weight (Bacciotti & Rosier, 2005). Let us finally mention that homogeneity with respect to an Euler vector field defined by a generalized weight $\mathbf{r}$ is usually simply referred to as $\mathbf{r}$-homogeneity.

[1] See (Teschl, 2012), Chapter 6 for details.