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Dynamical analysis of quantum linear systems driven by multi-channel multi-photon states[☆]

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ABSTRACT

In this paper, we investigate the dynamics of quantum linear systems where the input signals are multi-channel multi-photon states, namely states determined by a definite number of photons superposed in multiple input channels. In contrast to most existing studies on separable input states in the literature, we allow the existence of quantum correlation (for example quantum entanglement) in these multi-channel multi-photon input states. Due to the prevalence of quantum correlations in the quantum regime, the results presented in this paper are very general. Moreover, the multi-channel multi-photon states studied here are reasonably mathematically tractable. Three types of multi-photon states are considered: (1) m photons superposed among m channels, (2) N photons superposed among m channels where $N \geq m$, and (3) N photons superposed among m channels where N is an arbitrary positive integer. Formulae for intensities and states of output fields are derived. Examples are used to demonstrate the effectiveness of the results.

1. Introduction

Dynamical response analysis is an essential ingredient of control engineering, and is also the basis of controller design. For example, impulse response, step response, and frequency response are standard materials in modern control textbooks, see, e.g., Anderson and Moore (1979) and Kwakernaak and Sivan (1972). Fluctuation analysis of a dynamical system driven by white noise underlies the celebrated Kalman filter and linear quadratic Gaussian (LQG) control. Likewise, in the quantum regime, the response of quantum linear systems to quantum Gaussian white noise is the basis of quantum filtering and measurement-based feedback control, see, e.g., Belavkin (1993), Dong and Petersen (2010), Guta and Yamamoto (2016), Wiseman and Milburn (2010) and Zhang, Wu, Liu, Li, and Tarn (2012) and references therein.

In addition to quantum Gaussian noise commonly dealt with in quantum optical laboratories, in recent years, highly nonclassical signals such as single-photon states, multi-photon states, and Schrödinger's cat states have been attracting growing interest due to their promising applications in quantum information technology. Roughly speaking, an ℓ-photon state of a light beam means that the light field contains *exactly* ℓ photons. In this paper, we are concerned with continuous-mode ℓ-photon states, that is, these photons are specified by their frequency profiles centered at the carrier frequency of the light field. Continuous-mode single- and multi-photon states have found important applications in quantum computing, quantum communication, and quantum metrology, see, e.g., Baragiola, Cook, Branczyk, and Combes (2012), Bartley, Donati, Spring, Jin, Barbieri, Datta, and et al. (2012), Gheri, Ellinger, Pellizzari, and Zoller (1998), Gough, James, Nurdin, and Combes (2012), Loudon (2000), Milburn (2008), Ou (2007), Yamamoto and James (2014) and Zhang and James (2013), and Nysteen, Kristensen, McCutcheon, Kaer, and Mazrk (2015).

In the quantum control community, the response of quantum systems to single- and multi-photon states has been studied in the past few years. The phenomenon of cross phase shift on a coherent signal induced by a single photon pulse was investigated in Munro, Nemoto, and Milburn (2010). Gough et al. derived quantum filters for Markov quantum systems driven by single-photon states or Schrödinger's cat states (Gough et al., 2012). The theory in Gough et al. (2012) was applied to the study of phase modulation in Carvalho, Hush, and James (2012). Quantum master equations for an arbitrary quantum system driven by multi-photon states were derived in Baragiola et al. (2012). Quantum filters for multi-photon states were derived in Song, Zhang, and Xi (2016), for both homodyne detection and photodetection measurements. Numerical simulations carried out in Song et al. (2016) for a two-level system driven by a 2-photon state revealed interesting and

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complicated nonlinear behavior in the photon–atom interaction. When a two-level atom, initialized in the ground state, is driven by a single photon, the exact form of the output field state was derived in Pan, Zhang, and James (2016). More discussions can be found in, e.g., Fan, Kocabas, and Shen (2010) and Nysteen et al. (2015) and references therein.

In Zhang and James (2013), an analytic expression of the output field state of a quantum linear system driven by a single-photon state was derived. The research initialized in Zhang and James (2013) was continued and extended in Zhang (2014), where multi-photon states were considered. Unfortunately, the multi-photon states studied in Zhang (2014) are either with very limited quantum correlation or mathematically formidable. More specifically, the multi-photon states defined in Zhang (2014, Eqs. (23) and (41)) are separable states, i.e., there exists no entanglement among the channels. A class of photon-Gaussian states was defined in Zhang (2014, Eq. (34)). On the one hand, this class of states appears mathematically complicated. On the other hand, because each pulse shape is indexed by three parameters only, the feature of the multi-channel entanglement is unclear. A class of multi-channel multi-photon states was defined in Zhang (2014, Eq. (43)), which, due to the presence of an m-fold product, looks rather complicated mathematically.

The purpose of this paper is to provide a direct study of the dynamical response of quantum linear systems to initially entangled multi-channel multi-photon states. Unlike those separable states studied in Zhang and James (2013) and Zhang (2014), the multi-channel multi-photon states proposed in this paper are able to capture the entanglement among channels. Examples presented in this paper demonstrate that these types of multi-channel multi-photon states can be easily processed by quantum linear systems. Furthermore, the proposed multi-channel multi-photon states are very general as they contain many types of multi-channel multi-photon states as special cases, see, e.g., Brecht, Reddy, Silberhorn, and Raymer (2015), Loudon (2000, Chapter 6) and Rohde, Mauerer, and Silberhorn (2007). Finally, these states are mathematically more tractable than those in Zhang (2014, Eqs. (34) and (43)). Therefore, the study carried out in this paper is more relevant to quantum linear feedback networks and control.

Three types of multi-channel multi-photon states are studied in this paper. Case (1): m photons are superposed among m channels. Specifically, the m-channel m-photon states are defined in Section 3.1. When the underlying quantum linear system is passive, an analytic expression of the output intensity is presented in Section 3.2, see Theorem 1. Moreover, the steady-state output field state is investigated in Sections 3.3 and 3.4, see Theorems 2 and 3. When the underlying quantum linear system is non-passive, the steady-state output field state is no longer an m-channel m-photon state, an explicit form of the output field state is given in Section 3.5, see Theorem 4. Case (2): N photons are superposed among m channels where $N \geq m$. For this case, we assume the underlying quantum linear system is passive. The analytic expressions of the output field state are derived, see Theorems 5 and 6. Case (3): N photons are superposed among m channels, where N is an arbitrary positive integer. Specifically, a class of m-channel N-photon states are first presented in Section 5.1, then in Section 5.2, the steady-state output field state of a quantum linear passive system driven by an m-channel N-photon input state is derived, see Theorem 7.

Notation. The imaginary unit $\sqrt{-1}$ is denoted by i. Given a column vector of complex numbers or operators $x = [x_1 \ \cdots \ x_k]^T$, define a column vector $x^\# \triangleq [x_1^* \ \cdots \ x_k^*]^T$, where the superscript "$*$" stands for complex conjugation of a complex number or Hilbert space adjoint of an operator. Define a row vector $x^\dagger \triangleq (x^\#)^T = [x_1^* \ \cdots \ x_k^*]$. Define a doubled-up column vector $\breve{x} \triangleq [x^T \ x^\dagger]^T$. Let I_k be an identity matrix and 0_k a zero square matrix, both of dimension k. Denote $J_k = \mathrm{diag}(I_k, -I_k)$. Given a matrix $X \in \mathbb{C}^{2j\times 2k}$, define $X^\flat \triangleq J_k X^\dagger J_j$. Given a matrix A, let A^{jk} denote the entry on the jth row and kth column. Let m be the number of input channels. Let n be the number of degrees of freedom of a given quantum linear system, namely the number of quantum harmonic oscillators. The ket $|\phi\rangle$ denotes the initial state of the system of interest, and $|0\rangle$ stands for the vacuum state of free fields. The convolution of two functions f and g is denoted as $f \circledast g$. Given two matrices $U, V \in \mathbb{C}^{r\times k}$, define a doubled-up matrix $\Delta(U, V) \triangleq [U \ V; V^\# \ U^\#]$. Given two operators $\mathcal{A}$ and $\mathcal{B}$, their commutator is defined to be $[\mathcal{A}, \mathcal{B}] \triangleq \mathcal{AB} - \mathcal{BA}$. The Kronecker delta function is denoted by δ_{jk}, whereas the Dirac delta function is denoted by $\delta(t)$. The m-fold integral $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m$ is sometimes denoted by $\int d\overrightarrow{t}$. Given a function $f(t)$ in the time domain, define its *two-sided* Laplace transform (Sogo, 2010 Eq. (13)) to be $F[s] \equiv \mathscr{L}_b\{f(t)\}(s) \triangleq \int_{-\infty}^{\infty} e^{-st} f(t)dt$. The m dimensional Fourier transform of an m-variable function $f(t_1, \ldots, t_m)$ is, (Bracewell, 1999),

$$f(\mathrm{i}\omega_1, \ldots, \mathrm{i}\omega_m) \triangleq \frac{1}{(2\pi)^{m/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m e^{-\mathrm{i}(\omega_1 t_1 + \cdots + \omega_m t_m)} f(t_1, \ldots, t_m). \quad (1)$$

We set $\hbar = 1$ throughout this paper.

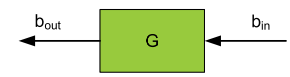

Fig. 1. Quantum linear system G with input b_{in} and output b_{out}.

2. Preliminaries

In this section, quantum linear systems are briefly introduced; more discussions can be found in, e.g., Gardiner and Zoller (2004), Gough and James (2009), James and Gough (2010), Petersen (2011), Walls and Milburn (2008), Wiseman and Milburn (2010), Yamamoto (2014), Zhang and James (2011), Zhang and James (2012) and Nurdin (2014). Some tensors and their associated operations are also discussed.

2.1. Quantum linear systems

A quantum linear system G is shown schematically in Fig. 1. In this model, the quantum linear system G consists of a collection of n (interacting) quantum harmonic oscillators represented by $a = [a_1 \ \cdots \ a_n]^T$. Here, a_j ($j = 1, \ldots, n$), defined on a Hilbert space $\mathfrak{H}$, is the annihilation operator of the jth quantum harmonic oscillator. The adjoint operator of a_j, denoted by a_j^*, is called a creation operator. These operators satisfy the following canonical commutation relations: $[a_j, a_k^*] = \delta_{jk}$, and $[a_j, a_k] = [a_j^*, a_k^*] = 0$, ($j, k = 1, \ldots, n$). The input fields are represented by a vector of annihilation operators $b_{\mathrm{in}}(t) = [b_{\mathrm{in},1}(t) \ \cdots \ b_{\mathrm{in},m}(t)]^T$; the entry $b_{\mathrm{in},j}(t)$ ($j = 1, \ldots, m$), defined on a Fock space $\mathfrak{F}$, is the annihilation operator for the jth input channel. The adjoint operator of $b_{\mathrm{in},j}(t)$, denoted by $b_{\mathrm{in},j}^*(t)$, is also called a creation operator. However, unlike a_j and a_k^*, the annihilation and creation operators for the input fields satisfy the following singular commutation relations, (Gardiner & Zoller, 2004; Gough, James, & Nurdin, 2010, Eq. (20)),

$$[b_{\mathrm{in},j}(t), b_{\mathrm{in},k}^*(r)] = \delta_{jk}\delta(t-r),$$
$$[b_{\mathrm{in},j}(t), b_{\mathrm{in},k}(r)] = [b_{\mathrm{in},j}^*(t), b_{\mathrm{in},k}^*(r)] = 0, \quad (2)$$

for $j, k = 1, \ldots, m$ and $\forall t, r \in \mathbb{R}$. Notice the presence of the Dirac delta function $\delta(t-r)$ in Eq. (2). Mathematically, it is often