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Brief paper

On optimization of stochastic max–min-plus-scaling systems—An approximation approach☆

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ABSTRACT

A large class of discrete-event and hybrid systems can be described by a max–min-plus-scaling (MMPS) model, i.e., a model in which the main operations are maximization, minimization, addition, and scalar multiplication. Accordingly, optimization of MMPS systems appears in different problems defined for discrete-event and hybrid systems. For a stochastic MMPS system, this optimization problem is computationally highly demanding as often numerical integration has to be used to compute the objective function. The aim of this paper is to decrease such computational complexity by applying an approximation method that is based on the moments of a random variable and that can be computed analytically.

1. Introduction

Stochastic max–min-plus-scaling (MMPS) systems construct a special class of stochastic discrete-event and hybrid systems, in which processing times and/or transportation times are stochastic quantities; in practice stochastic fluctuations of these times can, e.g., be caused by machine failure or depreciation (Olsder, Resing, de Vries, Keane, and Hooghiemstra, 1990). The system dynamics of an MMPS system are defined by MMPS expressions, i.e., expressions constructed using the operations maximization, minimization, addition, and multiplication by a scalar. In Necoara, De Schutter, van den Boom, and Hellendoorn (2008) it was shown that the class of MMPS systems encompasses other classes of discrete-event systems such as max-plus linear systems. Furthermore, it has been shown in Gorokhovik and Zorko (1994), Heemels, De Schutter, and Bemporad (2001), Ovchinnikov (2002) that MMPS systems are equivalent to a particular class of hybrid systems, namely continuous piecewise affine (PWA) systems.

In optimization problems for stochastic MMPS or continuous PWA systems, the objective function is often defined as the expected value of an MMPS or a continuous PWA function. Since, in general, there are no analytic expressions for such an expected value, the computation of the objective function in principle involves numerical integration, which is computationally complex and very time consuming. The aim of this paper is to develop an approximation method to compute the expected value of a stochastic MMPS or continuous PWA function with focus on reducing the computational complexity and the computation time. This approximation method is an extension of the method presented in Farahani, van den Boom, van der Weide, and De Schutter (2016), which is inspired by the relation between different types of vector norms, namely the p-norm and the ∞-norm and which in Farahani et al. (2016) has been only applied to max-plus linear systems with normally distributed disturbances. In Farahani et al. (2011), the method proposed in Farahani et al. (2016) has been applied in the context of model predictive control for stochastic MMPS systems and in Farahani, van den Boom, and De Schutter(2014), the approximation method has been extended to a more general class of distributions and an upper bound for the error of this method has been introduced.

The main contributions of the current paper are as follows: (1) proposing a solution for the optimization problem of stochastic MMPS systems using an approximation method that is based on moment-generating functions and is applicable to any distribution with finite moments; (2) discussing the error of the proposed approximation method and presenting finite upper bounds for the error caused by this approximation method. In the discussion of the general optimization problem of stochastic MMPS systems, we introduce two main applications of such systems, namely, the filtering problem and the reference tracking problem. To solve

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the optimization problem, we use the approximation method proposed in Farahani et al. (2014), which provides an upper bound for the expected value of a stochastic MMPS function and which can be used as a replacement of the objective function in the optimization problem. In the error discussion, besides presenting an upper bound, we show how different parameters in the approximation function may influence the error bounds.

2. Max–min-plus-scaling systems

A large class of discrete-event and hybrid systems can be described by a max–min-plus-scaling (MMPS) model.[1] These models are described using MMPS functions.

Definition 1 (*De Schutter and van den Boom, 2002*). A function $g : \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued MMPS function of the variables $x_1, \dots, x_n$ if there exist an index $i \in \{1, \dots, n\}$ and scalars $\alpha, \beta \in \mathbb{R}$ such that

$$g(x) = x_i|\alpha|\max(g_k(x), g_l(x))|\min(g_k(x), g_l(x))|$$
$$g_k(x) + g_l(x)|\beta g_k(x),$$

where | stands for "or" and g_k and g_l are scalar-valued MMPS functions.

Accordingly, for a vector-valued MMPS function g, each component of g is an MMPS function of the above form.

A state space representation of a stochastic MMPS system, in which noise and modeling errors are present, can be described as

$$x(k) = \mathcal{M}_x(x(k-1), u(k), \omega(k)) \quad (1)$$
$$y(k) = \mathcal{M}_y(x(k), u(k), \omega(k)) \quad (2)$$

where $\mathcal{M}_x$, $\mathcal{M}_y$ are MMPS functions, $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ is the system input, and $y(k) \in \mathbb{R}^s$ is the system output at time or event step k. We present both noise and modeling errors in a single framework using a vector $\omega(k)$, which is a vector of independent random variables with a given probability distribution.

The class of MMPS systems is equivalent to a particular class of hybrid systems, namely the class of continuous PWA systems (see Bemporad, Ferrari-Trecate, & Morari, 2000; Chua & Deng, 1988; Johansson, 2003; Leenaerts & van Bokhoven, 1998 for more details on PWA systems).

Proposition 2 (*Gorokhovik & Zorko, 1994; Ovchinnikov, 2002*). *Any MMPS function can be written as a continuous PWA function and vice versa.*

Moreover, any MMPS function can be written in a canonical form, as expressed in the following proposition.

Proposition 3 (*De Schutter and van den Boom, 2004*). *Any scalar-valued MMPS function g can be rewritten into the min–max canonical form $g(x) = \min_{i=1,\dots,K}\max_{j\in n_i}(\alpha_{ij}^T x + \beta_{ij})$ or into the max–min canonical form $g(x) = \max_{i=1,\dots,L}\min_{j\in m_i}(\gamma_{ij}^T x + \delta_{ij})$ for some integers K, L, non-empty subsets n_i and m_i of the index sets $\{1, 2, \dots, K\}$ and $\{1, 2 \dots, L\}$ respectively, real numbers β_{ij}, δ_{ij}, and vectors α_{ij}, γ_{ij}.*

Furthermore, the following proposition from Farahani et al. (2011, Corollary 5) shows that an MMPS function can be written as a difference of two convex functions.

[1] Note that generalized Lindley recursion models Borovkov (1984) and Whitt (1990) are special case of MMPS systems.

Proposition 4. *The function $g(x) = \max_{i=1,\dots,L}\min_{j=1,\dots,m_i} l_{ij}(x)$ where $l_{ij}(x) = \gamma_{ij}^T x + \xi_{ij}$ is an affine function in x, can be written as $g(x) = s(x) - r(x)$ where $s(\cdot)$ and $r(\cdot)$ are both convex functions defined as follows*

$$r(x) = -\sum_{i=1}^{L}\min_{j=1,\dots,m_i} l_{ij}(x) = \sum_{i=1}^{L}\max_{j=1,\dots,m_i}(-l_{ij}(x)) \quad (3)$$

$$s(x) = r(x) + \max_{i=1,\dots,L}\min_{j=1,\dots,m_i} l_{ij}(x)$$
$$= \max_{l=1,\dots,L}\ \max_{(j_1,\dots,j_{i-1},j_{i+1},\dots,j_L)\in\mathcal{C}(m_1,\dots,m_{i-1},m_{i+1},\dots,m_L)} \sum_{\substack{i'=1\\ i'\neq i}}^{L}(-l_{i'j_{i'}}(x)). \quad (4)$$

The last equality is obtained using the distributive property of addition w.r.t. maximization in which for some integers $L, m_1, \dots, m_L$, the set $\mathcal{C}(m_1, \dots, m_L)$ is defined as $\mathcal{C}(m_1, \dots, m_L) = \{(q_1, \dots, q_L)|q_k \in \{1, 2, \dots, m_k\}, k = 1, \dots, L\}$.

3. Optimization of stochastic MMPS systems

We consider minimization of a stochastic MMPS or continuous PWA function with a random vector ω that has a given probability density function. The class of minimization problems under consideration[2] can be formulated as

$$\min_{u\in\mathbb{R}^n}\ \mathbb{E}_\omega[F(u, \omega)]$$
$$\text{s.t.}\quad G(u) \leq 0 \quad (5)$$

where $\mathbb{E}_\omega[\cdot]$ is the expected value operator with respect to ω, F is a scalar MMPS function of u and ω, and G is a convex function of u specifying the input constraints. In order to solve the optimization problem (5), we need to determine the expected value of an MMPS function. One possible approach is numerical integration using the available methods. However, numerical integration is in general both cumbersome and time-consuming, and it becomes even more complicated as the probability density function of ω becomes more complex. Therefore, it is desired to find an alternative approach that is more efficient than numerical integration.

First, we apply Proposition 4 to rewrite the objective function $\tilde{J}(u) = \mathbb{E}_\omega[F(u, \omega)]$ as a difference of two convex functions:

$$\tilde{J}(u) = \mathbb{E}_\omega[F(u, \omega)] = \mathbb{E}_\omega[s(u, \omega) - r(u, \omega)]$$
$$= \mathbb{E}_\omega[s(u, \omega)] - \mathbb{E}_\omega[r(u, \omega)] \quad (6)$$

where $s(\cdot, \cdot)$ and $r(\cdot, \cdot)$ are defined as given in Proposition 4, and where the last equality stems from the fact that $\mathbb{E}[\cdot]$ is a linear operator. Note that $\tilde{J}(u)$ in (6) results in a non-convex optimization problem. To solve the optimization problem (5), it is only left to compute the expected values in (6). Note that $s(u, \omega)$ and $r(u, \omega)$ both consist of a maximization of set of affine terms. Therefore, our aim is to find an efficient way to compute the following general expression:

$$\mathbb{E}[\max_{j=1,\dots,n}(\xi_j + \gamma_j^T\omega)] \quad (7)$$

where $\xi_j \in \mathbb{R}$, $\gamma_j \in \mathbb{R}^{n_\omega}$ is a scaling factor, and $\omega \in \mathbb{R}^{n_\omega}$ is a vector of independent random variables with a given probability distribution. Note that by assumption $\xi_j = \alpha_j + \beta_j u$, for $\alpha_j \in \mathbb{R}$ and $\beta_j \in \mathbb{R}^m$ but that the dependence of ξ_j on u is dropped in the rest of the paper for brevity. Next, we present two popular cases in which the optimization of stochastic MMPS functions appears.

[2] This class consists of one-stage horizon and receding horizon (model predictive control) optimization problems and the class of control problems with static open-loop inputs.