Automatica 83 (2017) 58–64

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Properties and stability analysis of discrete-time negative imaginary systems ☆

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ARTICLE INFO

Article history:
Received 1 February 2016
Received in revised form 16 February 2017
Accepted 19 April 2017

Keywords:
Discrete-time systems
Negative imaginary lemma
Positive feedback
Internal stability

ABSTRACT

This paper is concerned with discrete-time negative imaginary (DT-NI) functions. First, a new definition of DT-NI functions is introduced. Then, by means of the relations between discrete-time positive real and DT-NI functions, two different versions of DT-NI lemmas are established to characterize the DT-NI properties based on state-space realizations. Also, a necessary and sufficient condition is presented to guarantee the internal stability of positive feedback interconnected DT-NI systems. Meanwhile, some other properties of DT-NI functions are studied. Several numerical examples are presented to illustrate the main results of this paper. Compared to the previous results, our results remove the symmetric assumption in rational case.

1. Introduction

The concept of positive real (PR) functions originated in network theory (Anderson & Vongpanitlerd, 1973). PR systems have obtained great achievements both in theory and in practice (Brogliato, Lozano, Maschke & Egeland 2007). One major limitation of PR functions is that their relative degree must be zero or one (Brogliato et al., 2007; Xiong, Petersen & Lanzon 2010). The theory of negative imaginary (NI) systems, who allowed a maximum relative degree of two (Lanzon & Petersen, 2008; Mabrok, Kallapur, Petersen & Lanzon 2014; Xiong et al., 2010), has appeared as a useful complement to PR theory. Since the NI theory was first proposed in Lanzon and Petersen (2008), a bunch of extensive study has arisen from the theory of NI systems to the application of NI theory, e.g., see Cai and Hagen (2010), Liu and Xiong (2016b), Mabrok et al. (2014), Patra and Lanzon (2011) and Petersen and Lanzon (2010). In particular, the internal stability results of positive feedback interconnected systems with NI response play an important role in engineering applications, see Lanzon and Petersen (2008), Mabrok et al. (2014) and Xiong et al. (2010).

It is noteworthy that all the present theory and applications of NI systems focus on the study of continuous-time (CT) systems except Ferrante, Lanzon & Ntogramatzidis (2014). In this paper, we are interested in presenting a similar development for discrete-time (DT) real-rational proper systems without the symmetric restriction. One should realize that this work is important in practice for the following reasons: (1) Almost all modern control schemes are digital signals in nature (Jiang, 1993). To analyse the closed-loop systems stability or properties of these control schemes, one should discretize the systems via a suitable sampling with a zero-order hold device (Jiang, 1993). This sample procedure leads to DT systems. (2) Although the generalized concept of DT-NI functions via z-domain has been proposed in Ferrante et al. (2014) to allow the DT-NI functions to be non-rational, all the transfer function matrices in Ferrante et al. (2014) are limited to be symmetric.

As is well-known, the bilinear transformation $s = \frac{z-1}{z+1}$ maps the open left half plane for CT systems to the open unit disc for DT systems (Anderson, Hitz & Diem, 1974; Ober & Montgomery-Smith, 1990). Under this bilinear transformation, a continuous-time positive real (CT-PR) transfer function $F(s)$ with $F(\infty) < \infty$ is transformed into a discrete-time positive real (DT-PR) transfer function $F(z)$ with $F(-1) < \infty$ and vice versa (Anderson et al., 1974; Hitz & Anderson, 1969). Also, the DT-PR lemma in Hitz and Anderson (1969) was derived by using this transformation. Therefore, our main techniques to handle the properties of DT-NI transfer functions in this paper are based on the bilinear transformation. Much as the continuous-time negative imaginary (CT-NI) systems can be defined in terms of their properties on the purely imaginary axis, the DT-NI systems can be related to their behaviours on the unit circle.

The contributions of this paper are as follows: (1) A new definition for DT-NI transfer function matrices that may be non-symmetric is introduced; (2) Under different assumptions, the

☆ This work was supported by National Natural Science Foundation of China under Grant 61374026. The material in this paper was not presented at any conference. This paper was recommended for publication in the revised form by Associate Editor Delin Chu under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2017.05.006
0005-1098/© 2017 Published by Elsevier Ltd.

relations between DT-PR and DT-NI functions are studied; (3) Two different DT-NI lemmas are derived by removing the symmetric assumption; (4) By checking the loop gain at $z = 1$ of the positive feedback system, a necessary and sufficient condition is derived for the internal stability of interconnected DT-NI systems. Compared to the results in Ferrante et al. (2014), our main contribution in this paper is that the real-rational transfer function matrix is allowed to be non-symmetric, that also develops the results on the real-rational DT-NI transfer function matrices. Meanwhile, a different version of DT-NI lemma is provided.

The rest of the paper is organized as follows. Section 2 provides the basic concept and some useful properties for DT-NI systems. Section 3 states the new relations between DT-PR and DT-NI functions. Two DT-NI lemmas are presented in Section 4. Section 5 presents the internal stability of positive feedback interconnected systems. Section 6 concludes the paper.

Notation: $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote the sets of $m\times n$ real matrices and real-rational proper transfer function matrices, respectively. A^T, A^* and $\bar{A}$ denote the transpose, the complex conjugate transpose and the complex conjugate of a complex matrix A, respectively. $\bar{\lambda}$ denotes the maximum eigenvalue for a square complex matrix with only real eigenvalues. $A > (\geq) 0$ denotes a symmetric positive (semi-)definite matrix. I denotes any identity matrix with compatible dimensions.

2. Discrete-time negative imaginary transfer functions

In this section, a new definition of DT-NI transfer function matrices is proposed, and some useful properties of such functions are studied.

Lemma 1 (*Hitz and Anderson, 1969*). *A square matrix $F(z)$ whose elements are real-rational functions analytic in $|z| > 1$ is DT-PR if, and only if, it satisfies all the following conditions*

(1) *poles of elements of $F(z)$ on $|z| = 1$ are simple;*
(2) *$F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all real θ at which $F(e^{j\theta})$ exists;*
(3) *if $z_0 = e^{j\theta_0}$, θ_0 is real, is a pole of an element of $F(z)$, and if K_0 is the residue matrix of $F(z)$ at z_0, then the matrix $e^{-j\theta_0}K_0$ is a nonnegative definite Hermitian.*

Remark 1. Conditions 1–3 of Lemma 1 can be replaced by (a) $F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$, with $e^{j\theta}$ not a pole of any element of $F(z)$; (b) If $z_0 = e^{j\theta_0}$, $\theta_0 \in [0, 2\pi]$, is a pole of an element of $F(z)$, then it is a simple pole (that is the poles of $F(z)$ on the unit circle, $|z| = 1$, are simple), and the corresponding residue matrix $K_0 = \lim_{z\to z_0}(z - z_0)F(z)$ satisfies that $e^{-j\theta_0}K_0$ is a nonnegative definite Hermitian.

By analogy with the CT case, we now present a new definition of DT-NI transfer function matrices.

Definition 1. A square real-rational proper transfer function matrix $G(z)$ is called DT-NI if

(1) $G(z)$ has no poles in $|z| > 1$;
(2) $j[G(e^{j\theta}) - G^*(e^{j\theta})] \geq 0$ for all $\theta \in (0, \pi)$ except values of θ where $e^{j\theta}$ is a pole of $G(z)$;
(3) if $z_0 = e^{j\theta_0}$, $\theta \in (0, \pi)$, is a pole of $G(z)$, then it is a simple pole and the corresponding residue matrix $K = \lim_{z\to z_0}(z - z_0)jG(z)$ satisfies that $e^{-j\theta_0}K$ is a positive semidefinite Hermitian;
(4) if $z = 1$ is a pole of $G(z)$, then $\lim_{z\to 1}(z - 1)^2G(z)$ is a positive semidefinite Hermitian, and $\lim_{z\to 1}(z - 1)^mG(z) = 0$ for all $m \geq 3$;
(5) if $z = -1$ is a pole of $G(z)$, then $\lim_{z\to -1}(z + 1)^2G(z)$ is a negative semidefinite Hermitian, and $\lim_{z\to -1}(z+1)^mG(z) = 0$ for all $m \geq 3$.

In order to analyse the properties of DT-NI systems, we define the following matrices for a given DT-NI transfer function matrix $G(z)$:

$$A_2 = \lim_{z\to 1}(z-1)^2G(z), \qquad A_1 = \lim_{z\to 1}(z-1)\left(G(z) - \frac{A_2}{(z-1)^2}\right),$$
$$C_2 = \lim_{z\to -1}(z+1)^2G(z), \qquad C_1 = \lim_{z\to -1}(z+1)\left(G(z) - \frac{C_2}{(z+1)^2}\right).$$

According to Conditions 4 and 5 in Definition 1, $A_2 = A_2^* \geq 0$ and $C_2 = C_2^* \leq 0$.

Remark 2. When $G(z)$ is real-rational non-proper, it means that $G(z)$ has poles in $|z| > 1$, which does not satisfy Condition 1 of Definition 1. So, the present definition of DT-NI functions focuses on the proper function. For example, consider $G(z) = z$. $G(z)$ has a simple pole in $|z| > 1$ and $j[G(e^{j\theta}) - G^*(e^{j\theta})] = -2\sin\theta \leq 0$ for all $\theta \in (0, \pi)$, which imply $G(z)$ is not DT-NI.

Remark 3. The difference between Ferrante et al. (2014, Lemma 11) and Definition 1 in this paper is that the transfer function matrices in Ferrante et al. (2014) are restricted to be symmetric, so that it requires that $A_1 \geq A_2$ and $C_1 \geq -C_2$ in Ferrante et al. (2014, Lemma 11), while Definition 1 in this paper does not have those restrictions.

Two useful lemmas are given as follows.

Lemma 2 (*Xiong et al., 2010*). *If $A = A^* \geq 0$, then $\bar{A} = \bar{A}^* \geq 0$.*

Lemma 3 (*Liu and Xiong, 2016a*). *A CT-NI transfer function matrix $G(s)$ transforms into a DT-NI transfer function matrix $G(z)$ by the bilinear transformation $s = \frac{z-1}{z+1}$. Conversely, a DT-NI transfer function matrix $G(z)$ transforms into a CT-NI transfer function matrix $G(s)$ by the bilinear transformation $z = \frac{1+s}{1-s}$.*

Then, we have the following result, which states one important property of DT-NI systems.

Lemma 4. *Given a square real-rational proper DT-NI transfer function matrix $G(z)$. Then, $A_1 + A_1^T \geq 0$, and $C_1 + C_1^T \geq 0$ hold.*

Proof. Since $G(z)$ is DT-NI, it follows that $G(z)$ has at most a double pole at 1 and -1. When $G(z)$ has no poles at 1 and -1, one has that $A_1 = 0$ and $C_1 = 0$, and hence $A_1 + A_1^T = 0$, and $C_1 + C_1^T = 0$.

Now, consider the case when $G(z)$ has poles at 1. Similar to the minor decomposition theory of CT case, we can write $G(z)$ in the form

$$G(z) = G_1(z) + \frac{A_1}{z-1} + \frac{A_2}{(z-1)^2}, \tag{1}$$

where $G_1(z)$ has no poles at 1. By means of the bilinear transformation

$$s = \frac{z-1}{z+1},\ z = \frac{1+s}{1-s}, \tag{2}$$

Eq. (1) transforms into

$$\begin{aligned} G(s) &= G_1\left(\frac{1+s}{1-s}\right) + \frac{A_1}{\frac{1+s}{1-s} - 1} + \frac{A_2}{(\frac{1+s}{1-s} - 1)^2} \\ &= G_1\left(\frac{1+s}{1-s}\right) - \frac{A_1}{2} + \frac{A_2}{4} + \frac{A_1 - A_2}{2s} + \frac{A_2}{4s^2}, \end{aligned}$$

where $G_1(\frac{1+s}{1-s}) - \frac{A_1}{2} + \frac{A_2}{4}$ has no poles at $s = 0$. It follows from Lemma 3 that $G(s)$ is CT-NI. $z = 1$ is a pole of $G(z)$ iff $s = 0$ is a pole of $G(s)$. According to Lemma 3 in Mabrok et al. (2014) and Lemma 2 in Liu and Xiong (2016b), we have $\frac{A_1-A_2}{2} + (\frac{A_1-A_2}{2})^T \geq 0$. It follows that $A_1 + A_1^T \geq A_2 + A_2^T \geq 0$, that is $A_1 + A_1^T \geq 0$.