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Brief paper

Optimal tradeoff between instantaneous and delayed neighbor information in consensus algorithms[☆]

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ABSTRACT

We consider a distributed consensus problem over a network, where at each time instant every node receives two pieces of information from disjoint neighboring sets: a weighted average of current states of neighbors from a primary network, and a weighted average of one-hop delayed states of neighbors from a secondary network. The proposed algorithm makes each node update its state to a weighted average of these individual averages. We show that convergence to consensus is guaranteed with non-trivial weights. We also present an explicit formula for the weights allocated to each piece of the information for the optimal rate of convergence, when the secondary network is the complement of the primary network. Finally numerical examples are given to explore the case when the neighbor sets of the agents do not cover the whole network.

1. Introduction

Recent years have witnessed great advances in the development of distributed algorithms for multi-agent systems. One benchmark problem is the design of distributed consensus algorithms that aim at driving a group of agents to reach an agreement on a variable of interest (Jadbabaie, Lin, & Morse, 2003; Moreau, 2005; Olfati-Saber & Murray, 2007). Along this research line, the impact of directed communication topologies, high-order dynamics, nonlinear interactions has been considered (Lin, Francis, & Maggiore, 2007; Liu, Slotines, & Barabasi, 2011; Wieland, Kim, & Allgower, 2011) and fruitful results have been obtained on formation control, coverage control, and network controllability (Cortes, Martinez, Karatas, & Bullo, 2003; Ren & Atkins, 2007; Scardovi & Sepulchre, 2009; Tanner, Jadbabaie, & Pappas, 2007).

In wireless communication networks, the information exchange between agents can sometimes be affected by time delays, which can have great impact on system performance. In the study of multi-agent systems, there have been continuing efforts on disclosing the effect of time delays in the process of reaching an agreement (Olfati-Saber & Murray, 2004). In Blondel, Hendrickx, Olshevsky, and Tsitsiklis (2005); Moreau (2004), the authors showed that consensus algorithms are robust to arbitrary bounded communication delays in both continuous-time and discrete-time settings. These results were extended to the case with unbounded time-varying coupling delays in Liu, Lu, and Chen (2010). Time-domain and frequency-domain approaches have been adopted to derive convergence conditions in the presence of input delays (Tian & Liu, 2009). While negative impact of time delays on the system performance was studied in the literature above, there are also research efforts making use of delayed information to accelerate the convergence of consensus algorithms. By introducing memory for each node, it has been shown that the convergence process of the consensus algorithm can speed up (Oreshkin, Coates, & Rabbat, 2010; Sarlette, 2014). In the voter model, where each voter is equipped with an individual inertia to change their opinion depending on the persistence time of a voter's current opinion, this can counter-intuitively lead to faster consensus (Stark, Tessone, & Schweitzer, 2008). In Jin and Murray (2006), a multi-hop relay protocol has been proposed for fast consensus in a network where an agent can send not only its own state but also a collection of its instantaneous neighbors' states.

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Motivated by that network nodes often have multiple radio interfaces in practice, we consider in this paper a scenario with two networks describing the interactions between nodes. The messages exchanged in the primary network are received immediately, while in the secondary network the messages are received with a one-hop delay. For such a prototypical setup, we ask under what circumstances the consensus protocol converges. In particular, we consider that every node receives two pieces of information from two disjoint neighboring sets in the primary and secondary networks: a weighted average of current states of the primary network neighborhood and a weighted average of one-hop delayed states of the neighborhood of the secondary network. The tradeoff between current and delayed information is characterized by a parameter in the system update equation. We give conditions on this parameter to ensure the convergence of the algorithm and explore the optimal value of the parameter that leads to the fastest convergence rate when the two neighbor sets of each agent cover the whole network.

The organization of this paper is as follows. We first formulate the problem in Section 2. The convergence conditions are given in Section 3. The optimal selection on the tradeoff parameter is presented in Section 4. Numerical results are given in Section 5 to study possible extensions. Concluding remarks are given in Section 6.

2. Problem Statement

Consider a network consisting of N agents (nodes) indexed in the set $\mathcal{V} = \{1, 2, \dots, N\}$. Interactions between nodes are carried out through two networks: the primary and secondary networks, which are described by two simple undirected graphs without self-loops. The messages exchange between nodes in the primary network is instantaneous, while in the secondary network the messages are received with a one-hop delay.

Let the undirected graphs $\mathcal{G}^{(1)} = (\mathcal{V}, \mathcal{E}^{(1)})$ and $\mathcal{G}^{(2)} = (\mathcal{V}, \mathcal{E}^{(2)})$ denote the primary and secondary networks, respectively. Let $\mathcal{G} = \mathcal{G}^{(1)} \cup \mathcal{G}^{(2)} = (\mathcal{V}, \mathcal{E}^{(1)} \cup \mathcal{E}^{(2)})$ and suppose that $\mathcal{E}^{(1)} \cap \mathcal{E}^{(2)} = \emptyset$. Let $\mathcal{N}_i^{(1)} = \{j : \{i, j\} \in \mathcal{E}^{(1)}\}$ be the set of neighbors of agent i in $\mathcal{G}^{(1)}$ and $\mathcal{N}_i^{(2)}$ defined similarly. Assume that each edge $\{i, j\} \in \mathcal{E}^{(1)}$ is associated with a weight $w_{ji} > 0$, each edge $\{i, j\} \in \mathcal{E}^{(2)}$ has weight $w_{ji}^{\dagger} > 0$, and assume that self-weights $w_{ii}, w_{ii}^{\dagger}$, $i = 1, \dots, N$, are non-negative (not necessarily all positive, in contrast to the literature (Blondel et al., 2005; Jadbabaie et al., 2003; Xiao & Wang, 2006)). Assume that the weights w_{ij} and $w_{ij}^{\dagger}$ satisfy the following assumption.

Assumption 1. For all i, $\sum_{j \in \{i\} \cup \mathcal{N}_i^{(1)}} w_{ij} = 1$ and $\sum_{j \in \{i\} \cup \mathcal{N}_i^{(2)}} w_{ij}^{\dagger} = 1$.

Time is slotted as $t = 0, 1, 2, \dots$, and each node i holds a scalar value $x_i(t)$. At each time t, agent i has access to two aggregated values:

(i) The instantaneous weighted average from neighbor set $\mathcal{N}_i^{(1)}$ and itself in the primary network given by

$$\mathcal{A}_i(t) := \sum_{j \in \{i\} \cup \mathcal{N}_i^{(1)}} w_{ij} x_j(t).$$

(ii) The one-step delayed weighted average from neighbor set $\mathcal{N}_i^{(2)}$ and itself in the secondary network given by

$$\mathcal{A}_i^{\dagger}(t) := \sum_{j \in \{i\} \cup \mathcal{N}_i^{(2)}} w_{ij}^{\dagger} x_j(t-1).$$

Let $x(t) = [x_1(t), \dots, x_N(t)]^T$ and define $x(-1) = x(0)$. The aim of the network is to reach a consensus making use of the two pieces of information at each node, $\mathcal{A}_i(t)$ and $\mathcal{A}_i^{\dagger}(t)$. We propose the following simple algorithm that makes a tradeoff between the current and the delayed information exchange:

$$x_i(t+1) = (1-\beta)\mathcal{A}_i(t) + \beta \mathcal{A}_i^{\dagger}(t), \qquad (1)$$

where the parameter β is a constant weight given to the delayed information.

We aim to analyze the range of β for the convergence of Algorithm (1) and its optimal value leading to the fastest convergence rate for a given network $\mathcal{G}$. Algorithm (1) relates to the algorithm studied in Jin and Murray (2006), where each agent sends to its neighbors not only its own state but also a collection of its instantaneous neighbors' states. The collection of each agent's instantaneous neighbors' states can be regarded as one-hop delayed information for the receiver. The convergence speed is accelerated compared to the original system where each agent only makes use of its neighbors' information.

Remark 1. Here we assume that $\mathcal{A}_i(t)$ and $\mathcal{A}_i^{\dagger}(t)$ are the messages received at each node i, so node i can distinguish between $\mathcal{A}_i(t)$ and $\mathcal{A}_i^{\dagger}(t)$, while it cannot infer the value of $x_j(t)$ or $x_j(t-1)$ of a neighbor j in $\mathcal{N}_i^{(1)}$ or $\mathcal{N}_i^{(2)}$, respectively. Note that $\mathcal{A}_i(t)$ can be written as $\mathcal{A}_i(t) = x_i(t) + \sum_{j=\mathcal{N}_i^{(1)}} w_{ij}(x_j(t) - x_i(t))$ for all $i = 1, \dots, N$ and $\mathcal{A}_i^{\dagger}(t)$ can be written in a similar way. In such expressions $x_i(t)$ only provides a description of the state without assuming that it is known to node i. Therefore the nodes do not have to possess the values of their absolute states according to some global coordinate system and only relative or aggregated states can be communicated (Olfati-Saber & Murray, 2007).

3. Convergence conditions

In this section, convergence conditions for Algorithm (1) are given for the case when $\mathcal{G}$ is connected and then for the case when $\mathcal{G}^{(2)}$ is the complement of $\mathcal{G}^{(1)}$.

3.1. $\mathcal{G}$ is connected

Define $y(t) = [x^T(t)\, x^T(t-1)]^T$ with $y(0) = [x^T(0)\, x^T(-1)]^T$. Let $W_1 \in \mathbb{R}^{N \times N}$ with $[W_1]_{ii} = w_{ii}$, $[W_1]_{ij} = w_{ij}$ for $\{j, i\} \in \mathcal{E}^{(1)}$, and $[W_1]_{ij} = 0$ otherwise. Similarly $W_2 \in \mathbb{R}^{N \times N}$ is defined by $[W_2]_{ii} = w_{ii}^{\dagger}$, $[W_2]_{ij} = w_{ij}^{\dagger}$ for $\{j, i\} \in \mathcal{E}^{(2)}$, and $[W_2]_{ij} = 0$ otherwise. Let

$$\Phi(\beta) = \begin{bmatrix} (1-\beta)W_1 & \beta W_2 \\ I & \mathbf{0} \end{bmatrix},$$

where I and $\mathbf{0}$ are the identity matrix and zero matrix with compatible dimension. It is clear that W_1 and W_2 are stochastic matrices (Horn & Johnson, 1985) from Assumption 1. Algorithm (1) can be rewritten as

$$y(t+1) = \begin{bmatrix} (1-\beta)W_1 & \beta W_2 \\ I & \mathbf{0} \end{bmatrix} y(t) := \Phi(\beta) y(t). \qquad (2)$$

Similar to Theorem 1 in Xiao and Boyd (2004) and Theorem 1 in Johansson and Johansson (2008), it can be shown that the necessary and sufficient conditions for Algorithm (1) converging to the average of its initial condition are (C1) $\Phi(\beta)\mathbf{1} = \mathbf{1}$; (C2) $\boldsymbol{\alpha}^T \Phi(\beta) = \boldsymbol{\alpha}^T$ for vector $\boldsymbol{\alpha}^T = [\alpha_1 \mathbf{1}^T\ \alpha_2 \mathbf{1}^T]^T$ with α_1, α_2 satisfying $\alpha_1 + \alpha_2 = 1$; and (C3) $\rho(\Phi(\beta) - 1/N \mathbf{1}\boldsymbol{\alpha}^T) < 1$, where $\mathbf{1}$ is an all-one vector with compatible dimension and $\rho(\cdot)$ is the spectral radius of a matrix. If these three conditions are satisfied, then $\lim_{t\to\infty} \Phi(\beta)^t = 1/N \mathbf{1}\boldsymbol{\alpha}^T$. The conditions (C1)–(C3) are equivalent to the condition that one is a simple eigenvalue of $\Phi(\beta)$ with $\mathbf{1}$ and $\boldsymbol{\alpha}$ being its corresponding right and left eigenvectors, respectively, and all the other eigenvalues lie inside the unit circle.