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Brief paper

Time-varying feedback for regulation of normal-form nonlinear systems in *prescribed* finite time[☆]

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ABSTRACT

While non-smooth approaches (including sliding mode control) provide explicit feedback laws that ensure finite-time stabilization but in terminal time that depends on the initial condition, fixed-time optimal control with a terminal constraint ensures regulation in prescribed time but lacks the explicit character in the presence of nonlinearities and uncertainties. In this paper we present an alternative to these approaches, which, while lacking optimality, provides explicit time-varying feedback laws that achieve regulation in prescribed finite time, even in the presence of non-vanishing (though matched) uncertain nonlinearities. Our approach employs a scaling of the state by a function of time that grows unbounded towards the terminal time and is followed by a design of a controller that stabilizes the system in the scaled state representation, yielding regulation in prescribed finite time for the original state. The achieved robustness to right-hand-side disturbances is not accompanied by robustness to measurement noise, which is also absent from all controllers that are nonsmooth or discontinuous at the origin.

1. Introduction

Motivation. Regulation in finite time (Haimo, 1986) is commonly achieved using non-smooth feedback, including sliding mode control. However, regulation in *prescribed* finite time is a more demanding objective, which arises in missile guidance (Zarchan, 2007) and other applications. Two approaches to solving this problem are common—the classical (and elementary) *proportional navigation* feedback, which employs time-varying gains that go to infinity towards the terminal time, and optimal control with a terminal constraint, where such a dependency of the gains is implicit.

In this paper we present a systematic approach to regulation in prescribed finite time, which is inspired by PN for second-order missile model, but which we present for the general class of nonlinear systems in the "normal form" with a possibly non-vanishing uncertainty matched by control.

Literature on finite-time stabilization. Apart from classical sliding mode control, most finite time control results are built on the "Lyapunov differential inequality" introduced by Bhat and Bernstein (2000) and refined by Shen and Xia (2008) and Shen and Huang (2012). By using this inequality, together with other conditions, C^0 finite time feedback is presented for the double integrator by Bhat and Bernstein (1998) and for a class of planar systems by Qian and Li (2005). Homogeneous finite time local control for triangular systems and a certain class of nonlinear systems was developed by Hui, Haddad, and Bhat (2008), Hong (2002), Hong and Jiang (2006a); Hong, Wang, and Cheng (2006b). Huang, Lin, and Yang (2005) perform global finite-time stabilization of strict feedback systems; Polyakov and Poznyak (2009) present a sign function based (discontinuous) controller; Feng, Yu, and Man (2002) design a non-singular terminal sliding controller for robot systems; Shen and Huang (2009) present a global finite-time observer for globally Lipschitz systems; based on Implicit Lyapunov Functions (ILF) approach, finite-time and fixed-time stability analysis for a chain of integrators were presented in Li, Du, and Lin (2011), Polyakov, Efimov, and Perruquetti (2015), Wang, Li, and Shi (2014) and Wang and Xiao (2010) extended finite time control to consensus or containment of agents governed by single/double integrators.

The sophisticated technique of "adding power integration" introduced by Coron and Praly (1991) is employed by most authors including Huang et al. (2005), Huang et al. (2015), Li et al. (2011), Wang et al. (2014), and Wang, Song, Krstic, and Wen (2016).

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Most finite time controllers (Bhat & Bernstein, 1998, 2000; Feng et al., 2002; Hong, 2002; Hong et al., 2001; Hong & Jiang, 2006a; Hong et al., 2006b; Huang et al., 2015, 2005; Hui et al., 2008; Li et al., 2011; Miao & Xia, 2014; Qian & Li, 2005; Shen & Huang, 2009; Shen et al., 2015; Wang et al., 2014, 2016; Wang & Xiao, 2010) use fractional power feedback of the form $x^{\frac{l}{p}}$ (with p and l being some positive odd integers). Such control can only address constant unknown gains in second-order mechanical systems (Huang et al., 2015) or high-order systems with known control gains (Hong, 2002; Hong & Jiang, 2006a; Hong et al., 2006b; Huang et al., 2005; Polyakov et al., 2015; Shen & Huang, 2009).

Contributions of the paper. We introduce an entirely new methodology for solving finite-time regulation, with a prescribed regulation time, rather than a regulation time that depends on the initial condition (see Polyakov & Fridman, 2014) for differences between finite-time and fixed-time stability). We employ a *scaling of the state by a function that grows unbounded towards the terminal time* (somewhat akin to Seo et al., 2008), and then design a controller that stabilizes the system in the scaled state representation, yielding regulation in prescribed time for the original state. We develop our results for nonlinear systems diffeomorphically equivalent to the "normal form"

$$\begin{aligned}\dot{x}_i &= x_{i+1}, \quad i = 1, \ldots, n-1\\ \dot{x}_n &= f(x,t) + b(x,t)u,\end{aligned} \tag{1}$$

where $x = [x_1, \ldots, x_n]^T$ is the state, $u \in \mathbb{R}$ is control, and b, f are possibly uncertain and non-vanishing. Our result is limited to this class because non-vanishing uncertainties are impossible to reject in finite time unless they are matched by control. Since the stability proof for the class (1) for arbitrary n is rather complicated, we first present a result for the scalar case in Section 3 and then for the general case in Section 4. In addition to designs, in Section 2 we introduce new analysis tools in Lemma 1 and Corollary 1—time-varying counterparts of the lemmas by Bhat and Bernstein (2000).

The achieved robustness to right-hand-side disturbances is not accompanied by robustness to measurement noise, which is also absent from all controllers that are nonsmooth or discontinuous at the origin.

2. Assumptions and definitions

Assumption 1 (*Global Controllability*). For system (1) there exists a known $\underline{b} \neq 0$ (and w.l.o.g. $\underline{b} > 0$) such that $\underline{b} \leq |b(x,t)| < \infty$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$.

Assumption 2. (Bound on Matched but Possibly Nonvanishing Uncertainty) The nonlinearity f in (1) obeys

$$|f(x,t)| \leq d(t)\psi(x), \tag{2}$$

where $d(t)$ is a disturbance with an unknown bound

$$\|d\|_{[t_0,t]} := \sup_{\tau \in [t_0,t]} |d(\tau)|, \tag{3}$$

and $\psi(x) \geq 0$ is a known scalar-valued continuous function.

The basis of our fixed-time designs is the monotonically increasing function

$$\mu_1(t-t_0) = \frac{T}{T+t_0-t}, \quad t \in [t_0, t_0+T), \tag{4}$$

where $T > 0$, with the properties that $\mu_1(0) = 1$ and $\mu_1(T) = +\infty$. We introduce two *new* fixed-time stability definitions.

Definition 1 (*FT-ISS*). The system $\dot{x} = f(x,t,d)$ (of arbitrary dimensions of x and d) is said to be *fixed-time input-to-state stable in time T* (FT-ISS) if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that, for all $t \in [t_0, t_0+T)$,

$$|x(t)| \leq \beta\Big(|x_0|, \mu_1(t-t_0)-1\Big) + \gamma\left(\|d\|_{[t_0,t]}\right). \tag{5}$$

The function $\mu_1(t-t_0)-1 = (t-t_0)/(T+t_0-t)$ starts from zero at $t = t_0$ and grows monotonically to infinity as $t \to t_0+T$. Therefore, a system that is FT-ISS is, in particular, ISS, with the additional property that, in the absence of the disturbance d, it is fixed-time globally asymptotically stable in time T.

Definition 2 (*FT-ISS+C*). The system $\dot{x} = f(x,t,d)$ (of arbitrary dimensions of x and d) is said to be *fixed-time input-to-state stable in time T and convergent to zero* (FT-ISS+C) if there exist class $\mathcal{KL}$ functions β and β_f, and a class $\mathcal{K}$ function γ, such that, for all $t \in [t_0, t_0+T)$,

$$|x(t)| \leq \beta_f\left(\beta\Big(|x_0|, t-t_0\Big) + \gamma\left(\|d\|_{[t_0,t]}\right), \mu_1(t-t_0)-1\right). \tag{6}$$

Clearly a system that is FT-ISS+C is also FT-ISS, with the additional property that its state converges to zero in time T despite the presence of a disturbance.

Lemma 1. *Consider the function*

$$\mu(t-t_0) = \frac{T^{n+m}}{(T+t_0-t)^{n+m}} = \mu_1(t-t_0)^{n+m} \tag{7}$$

on $[t_0, t_0+T)$, with positive integers m, n. If a continuously differentiable function $V : [t_0, t_0+T) \to [0, +\infty)$ satisfies

$$\dot{V}(t) \leq -2k\mu(t-t_0)V(t) + \frac{\mu(t-t_0)}{4\lambda}d(t)^2 \tag{8}$$

for positive constants k and λ, then

$$V(t) \leq \zeta(t-t_0)^{2k}V(t_0) + \frac{\|d\|^2_{[t_0,t]}}{8k\lambda}, \quad \forall t \in [t_0, t_0+T), \tag{9}$$

where ζ is the monotonically decreasing (smooth "bump-like;" Fry & McManus (2002)) function

$$\zeta(t-t_0) = \exp^{\frac{T}{m+n-1}\left(1-\mu_1(t-t_0)^{m+n-1}\right)}, \tag{10}$$

with the properties that $\zeta(0) = 1$ and $\zeta(T) = 0$.

Proof. Solving the differential inequality (8) gives

$$\begin{aligned}V(t) \leq{}& \exp^{-2k\int_{t_0}^{t}\mu(\tau-t_0)d\tau}V(t_0)\\ &+ \frac{1}{4\lambda}\int_{t_0}^{t}\exp^{-2k\int_{\tau}^{t}\mu(s-t_0)ds}d(\tau)^2\mu(\tau-t_0)d\tau.\end{aligned} \tag{11}$$

We compute the second term on the right side of (11) to get

$$\begin{aligned}&\int_{t_0}^{t}\exp^{-2k\int_{\tau}^{t}\mu(s-t_0)ds}d(\tau)^2\mu(\tau-t_0)d\tau\\ &\leq \|d\|^2_{[t_0,t]}\int_{t_0}^{t}\exp^{2k\left(-\int_{t_0}^{t}\mu(s-t_0)ds+\int_{t_0}^{\tau}\mu(s-t_0)ds\right)}\mu(\tau-t_0)d\tau\\ &= \|d\|^2_{[t_0,t]}\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\\ &\quad\times\int_{t_0}^{t}\exp^{2k\int_{t_0}^{\tau}\mu(s-t_0)ds}d\left(\int_{t_0}^{\tau}\mu(s-t_0)ds\right)\\ &= \|d\|^2_{[t_0,t]}\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\frac{1}{2k}\exp^{2k\int_{t_0}^{\tau}\mu(s-t_0)ds}\Big|_{t_0}^{t}\\ &= \|d\|^2_{[t_0,t]}\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\frac{1}{2k}\left(\exp^{2k\int_{t_0}^{t}\mu(s-t_0)ds}-1\right)\\ &= \|d\|^2_{[t_0,t]}\frac{1}{2k}\left(1-\exp^{-2k\int_{t_0}^{t}\mu(s-t_0)ds}\right)\\ &\leq \frac{\|d\|^2_{[t_0,t]}}{2k}.\end{aligned} \tag{12}$$