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Staticization, its dynamic program and solution propagation☆

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ABSTRACT

Stationary-action formulations of dynamical systems are considered. Use of stationary-action formulations allows one to generate fundamental solutions for classes of two-point boundary-value problems (TPBVPs). One solves for stationary points of the payoff as a function of inputs rather than minimization/maximization, a task which is significantly different from that in optimal control problems. Both a dynamic programming principle (DPP) and a Hamilton–Jacobi partial differential equation (HJ PDE) are obtained for a class of problems subsuming the stationary-action formulation. Although convexity (or concavity) of the payoff may be lost as one propagates forward, stationary points continue to exist, and one must be able to use the DPP and/or HJ PDE to solve forward to such time horizons. In linear/quadratic models, this leads to a requirement for propagation of solutions of differential Riccati equations past finite escape times. Such propagation is also required in (nonlinear) n-body problem formulations where the potential is represented via semiconvex duality. The dynamic programming tools developed here are applicable.

1. Introduction

The classical approach to solution of energy-conserving dynamical systems is integration of Newton's second law. An alternative viewpoint is that a system evolves along a path which makes the action functional stationary, i.e., such that the first-order differential around the path is the zero element. This latter viewpoint appears particularly useful in some applications in modern physics, including gravitational systems where relativistic effects are non-negligible and systems in the quantum domain (cf. Feynman, 1948; Feynman, 1964; Gray & Taylor, 2007; Padmanabhan, 2010). Our interests are more pedestrian; the stationary-action formulation has recently been found to be quite useful for generation of fundamental solutions to two-point boundary-value problems (TPBVPs) for conservative dynamical systems. For sufficiently short time horizons, stationarity of the action typically corresponds to minimization of the action. That is, the stationary point is a global minimum of that action (cf., Dower & McEneaney, 2013; McEneaney & Dower, 2013, 2015). For longer time horizons, the stationary point is more typically a saddle.

As our motivating interest is in solution of TPBVPs for conservative dynamical systems, we note that this specifically includes mass–spring, wave equation and n-body problems (Dower & McEneaney, 2013; McEneaney & Dower, 2013, 2015). By appending a min-plus delta function terminal cost to the action functional, we obtain a fundamental solution object for such TPBVPs. Min-plus convolutions of this object with functionals associated to specific terminal conditions yield the solutions of the specific TPBVPs. As a change in the boundary data only requires convolution with a different functional, our object may best be termed a fundamental solution for TPBVPs, corresponding to the given time horizon. It is worth remarking that, further, one can populate the fundamental solution semigroup by convolving the fundamental solution with itself, enabling solution of the TPBVP for all strictly positive horizons.

As noted above, for sufficiently short time horizons, one may obtain the stationary action solution by minimization of the action functional, in which case it is obvious that the fundamental solution is derived from the value function for an optimal control formulation. However, for longer horizons, we must find the stationary point, and this requires a new set of tools. We define stationarity and value for such problems. Surprisingly, for a specific class of terminal costs, one may obtain a dynamic programming

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principle (DPP) for stationarity, where this is directly analogous to standard DPPs (for optimization). We do not look for the absolute weakest assumptions, but only a reasonable first-foray set. We also formally write the corresponding Hamilton–Jacobi partial differential equation (HJ PDE), and then obtain a verification result, which is also quite similar to that found for classical optimal control problems. We remark that a verification result implies that any solution of the HJ PDE in the specified class must be the value function. This validates the approach of solving a stationarity problem (and hence the related TPBVP if the stationarity problem is generated by such) by solving the associated HJ PDE problem.

In the mass–spring case (which appears in Section 3 as a motivating example), the stationary-action problem is linear–quadratic, and the HJ PDE reduces to a differential Riccati equation (DRE). By the above-noted verification result, we see that solution of the DRE yields solution of the stationarity problem, and any corresponding TPBVP. The wave equation (Dower & McEneaney, 2013) also yields a DRE, albeit infinite dimensional. The n-body problem may be reduced to a parameterized set of time-dependent DREs (McEneaney & Dower, 2013, 2015). We see that in all cases, solutions of DREs form a critical building block. Of course, DREs can exhibit finite escape times, and do so in these cases. In classical optimal control, one is not interested in propagation of the solution past such escape times. However, in stationarity problems, these may correspond to points where one loses convexity [concavity] of the payoff. Although the minimum [maximum] may go to $-\infty$ [$+\infty$], the stationary value may be well-defined and finite past such asymptotes, and one must propagate the solution beyond them. The DPP yields a means for propagation through escape times, and this will be indicated.

Although stationary action is the motivating problem class, the theory developed below is applicable to wider classes of problems, where one is seeking a stationary point. An obvious example is that of certain differential games. Extensions to stochastic cases appear possible as well, but are not considered here.

Section 2 contains relevant definitions. Section 3 presents a simple mass–spring TPBVP motivating example. Section 4 contains the main results—the DPP and HJ PDE verification theorem. Section 5 reduces to the linear/quadratic case, and indicates a means for propagation of DREs past escape times. Section 6 very briefly indicates some application areas.

2. Stationarity definitions

Recall that we are seeking stationary points of payoffs, which is unusual in comparison to the standard classes of problems in optimization. In analogy with the language for minimization and maximization, we will refer to the search for stationary points as staticization, with these points being statica (in analogy with minima/maxima) and a single such point being a staticum (in analogy with minimum/maximum). Prior to the development, we make the following definitions. Suppose $\mathcal{Y}$ is a generic normed vector space with $\mathcal{G}_\mathcal{Y} \subseteq \mathcal{Y}$, and suppose $F : \mathcal{G}_\mathcal{Y} \to \mathbb{R}$. We say $\bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\}$ if $\bar{y} \in \mathcal{G}_\mathcal{Y}$ and either

$$\limsup_{y \to \bar{y}, y \in \mathcal{G}_\mathcal{Y} \setminus \{\bar{y}\}} \frac{|F(y) - F(\bar{y})|}{|y - \bar{y}|} = 0, \tag{1}$$

or there exists $\delta > 0$ such that $\mathcal{G}_\mathcal{Y} \cap B_\delta(\bar{y}) = \{\bar{y}\}$ (where $B_\delta(\bar{y})$ denotes the ball of radius δ around $\bar{y}$). If $\operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} \neq \emptyset$, we define

$$\begin{aligned} \operatorname*{stat}_{y \in \mathcal{G}_\mathcal{Y}} F(y) &\doteq \operatorname{stat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} \\ &\doteq \left\{ F(\bar{y}) \,\middle|\, \bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} \right\}. \end{aligned} \tag{2}$$

If $\operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\} = \emptyset$, $\operatorname{stat}_{y \in \mathcal{G}_\mathcal{Y}} F(y)$ is undefined. Throughout, we will abuse notation by writing $\bar{y} = \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\}$ in the event that the argstat is the single point, $\{\bar{y}\}$, and similarly for stat.

In the case where $\mathcal{Y}$ is a Hilbert space, and $\mathcal{G}_\mathcal{Y} \subseteq \mathcal{Y}$ is an open set, $F : \mathcal{G}_\mathcal{Y} \to \mathbb{R}$ is Fréchet differentiable at $\bar{y} \in \mathcal{G}_\mathcal{Y}$ with Fréchet derivative $F_y(\bar{y}) \in \mathcal{Y}$ if

$$\lim_{v \to 0,\, \bar{y}+v \in \mathcal{G}_\mathcal{Y} \setminus \{\bar{y}\}} \frac{|F(\bar{y} + v) - F(\bar{y}) - \langle F_y(\bar{y}), v \rangle|}{|v|} = 0. \tag{3}$$

The following is immediate from the above definitions.

Lemma 1. *Suppose $\mathcal{Y}$ is a Hilbert space, with open set $\mathcal{G}_\mathcal{Y} \subseteq \mathcal{Y}$ and $\bar{y} \in \mathcal{G}_\mathcal{Y}$. Then, $\bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathcal{G}_\mathcal{Y}\}$ if and only if $F_y(\bar{y}) = 0$.*

3. Motivational examples

As indicated in the introduction, an important problem class which motivates this effort is that of TPBVPs for conservative systems.

3.1. A simple mass–spring problem

We first examine the classic one-dimensional mass–spring example in a substantial detail in order to provide motivation and insight. Although the problem is essentially trivial, it provides a nice means for obtaining a sense of the stationary action principle as a tool for understanding system dynamics and TPBVPs. Further, as remarked above, the stationary action viewpoint is the accepted viewpoint in modern physics (cf., Feynman, 1948; Feynman, 1964; Gray & Taylor, 2007; Padmanabhan, 2010), and as such, it will be ultimately necessary for advanced applications. It also provides exceptional computational advantages for difficult classes of problems, such as TPBVPs in the gravitational n-body case (McEneaney & Dower, 2013, 2015).

Remark 2. Although the mass–spring model has an analytically solvable form due to the quadratic potential, this potential is not physically reasonable (the potential approaches $+\infty$ as $|x| \to \infty$), and induces degeneracies, particularly at half-period times. Nonetheless, it is useful for building intuition.

Consider the mass–spring problem with mass, m, and spring-constant, K (typically given as $\ddot{\xi} = -(K/m)\xi$). The associated stationary action TPBVP payoff, $J^\infty : \hat{T} \times \mathbb{R} \times \mathcal{U}_\infty \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with $\hat{T} \doteq \{(s,t) \in \mathbb{R}^2 \,|\, 0 \le s \le t < \infty\}$ and $\mathcal{U}_\infty \doteq \mathcal{L}_2^{loc}(0,\infty)$, is given by

$$J^\infty(s,t,x,u,z) = \int_s^t \frac{m}{2} u^2(r) - \frac{K}{2} \xi^2(r)\, dr + \psi^\infty(\xi(t), z), \tag{4}$$

$$\text{where } \dot{\xi}(r) = u(r), \quad r \in (s,t), \qquad \xi(s) = x, \tag{5}$$

$$\psi^\infty(x,z) \doteq \begin{cases} 0 & \text{if } x = z, \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

Solution of this stationary-action problem will yield solution of the TPBVP given by dynamics $m\ddot{\xi}(r) = -\nabla V(\xi(r))$ with initial position $x \in \mathbb{R}^n$ and terminal position $z \in \mathbb{R}^n$ for the given duration t and given potential function (McEneaney & Dower, 2013, 2015), where in this example the potential is $V(x) = (K/2)x^2$.

The stationary action solution, u^*, is such that $J_u^\infty(s,t,x,u^*,z) = 0$, where J_u^∞ denotes the Fréchet derivative of J^∞ with respect to u as per (3). Here, we take $K = m = 1$. In McEneaney and Dower (2013, 2015), one notes that if $t - s < \pi/2$, then