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Brief paper

On turnpike and dissipativity properties of continuous-time optimal control problems[☆]

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ABSTRACT

This paper investigates the relations between three different properties, which are of importance in optimal control problems: dissipativity of the underlying dynamics with respect to a specific supply rate, optimal operation at steady state, and the turnpike property. We show in a continuous-time setting that if along optimal trajectories a strict dissipation inequality is satisfied, then this implies optimal operation at this steady state and the existence of a turnpike at the same steady state. Finally, we establish novel converse turnpike results, i.e., we show that the existence of a turnpike at a steady state implies optimal operation at this steady state and dissipativity with respect to this steady state. We draw upon a numerical example to illustrate our findings.

The notion of turnpike property of an optimal control problem (OCP) — introduced by Dorfman, Samuelson and Solow (1958) — is used to describe the phenomenon that in many finite-horizon OCPs the optimal solutions for different initial conditions approach a neighborhood of the best steady state, stay within this neighborhood for some time, and might leave this neighborhood towards the end of the optimization horizon. Turnpike phenomena have been observed in different types of OCPs: with/without terminal constraints (Carlson, Haurie & Leizarowitz, 1991; Clarke, 2013; Trélat & Zuazua, 2015) and with/without discounted cost functionals (Gurman & Ukhin, 2004; Würth, Rawlings & Marquardt, 2009; Zaslavski, 2014). Turnpikes have received widespread interest in the context of optimal control of economics (Carlson et al., 1991; Mckenzie, 1976). The works by Anderson and Kokotovic (1987), Rao and Mease (1999), Sahlodin and Barton (2015) and Wilde and Kokotovic (1972) show how turnpike phenomena can be used to approximate solutions of OCPs with long horizons appearing in applications.[1] Turnpikes also appear in OCPs arising in economic MPC formulations (Faulwasser & Bonvin, 2015a, b, 2017; Grüne, 2013; Würth et al., 2009). Recently, Grüne and Müller (2016) and Damm, Grüne, Stieler and Worthmann (2014) discussed different aspects of turnpike phenomena in a discrete-time setting with constraints and in a continuous-time setting without constraints (Trélat & Zuazua, 2015). Taking into account the large number of publications on turnpike phenomena, it is quite surprising that only very few works state a precise definition of turnpike properties, see Damm et al. (2014) and Zaslavski (2014). Often, turnpike results for specific OCPs are proven without a rigorous definition of the turnpike property itself (Carlson et al., 1991; Clarke, 2013; Gurman & Ukhin, 2004; Mckenzie, 1976). While such an approach simplifies the construction of many turnpike results, it hinders establishing *converse turnpike theorems*.

The main goal of this paper is to analyze the relation between three different properties that arise in the context of finite-horizon continuous-time OCPs: system dissipativity with respect to a specific supply rate (which depends on the cost function of the OCP), optimal operation at steady state, and the existence of a turnpike at that steady state. Recently, a related discrete-time analysis has been presented under the assumptions of local controllability of the turnpike and turnpike-like behavior of nearly optimal solutions (Grüne & Müller, 2016). The present paper takes a different

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[1] We remark that occasionally turnpike phenomena are denoted by varying names: Anderson and Kokotovic (1987) and Wilde and Kokotovic (1972) refer to turnpikes as a *dichotomy of optimal control problems*, while Rao and Mease (1999) uses the phrase *hypersensitive optimal control problems*.

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route by avoiding such assumptions in the continuous-time case. Its contributions are as follows: While the preliminary version of this paper (Faulwasser, Korda, Jones & Bonvin, 2014) discussed *state* turnpikes, we extend these results and provide a framework for the definition of different turnpike properties of OCPs, i.e., we suggest to distinguish *state, input–state*, and *extremal turnpikes* of OCPs. Our main contribution are novel converse turnpike results that require neither local controllability of the turnpike nor turnpike-like behavior of nearly optimal solutions as in (Grüne & Müller, 2016). In particular, we show that the existence of a turnpike implies optimal operation at steady state; we prove that exactness of turnpikes implies dissipativity, whereby exactness of a turnpike means that the optimal solutions are at the turnpike steady state for some parts of the optimization horizon; and we show that under mild local assumptions on the cost function of the OCP, the existence of a turnpike implies satisfaction of a strict dissipation inequality along optimal solutions.

The remainder of this paper is structured as follows: Section 1 introduces a formal definition of turnpike and dissipativity properties as well as the definition of optimal operation at steady state. Section 2 discusses implications of dissipativity. Section 3 investigates the relation between optimal operation at steady state and dissipativity, while Section 4 presents converse turnpike results. To demonstrate how some of our conditions can be verified, we draw upon the numerical example of a chemical reactor in Section 5.

1. Preliminaries and problem statement

We briefly recall the notions of optimal operation at steady state, dissipativity with respect to a steady state, and turnpike properties of OCPs.

1.1. Optimal steady–state operation

We consider the nonlinear system given by

$$\dot{x} = f(x, u), \quad x(0) = x_0, \tag{1}$$

where the states $x \in \mathbb{R}^{n_x}$ and the inputs $u \in \mathbb{R}^{n_u}$ are constrained to lie in the compact sets $\mathcal{X} \subset \mathbb{R}^{n_x}$ and $\mathcal{U} \subset \mathbb{R}^{n_u}$. We assume that the vector field $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ is Lipschitz on $\mathcal{X} \times \mathcal{U}$. A solution to (1), starting at x_0 at time 0, driven by the input $u : [0, \infty) \to \mathcal{U}$, is denoted as $x(\cdot, x_0, u(\cdot))$.

Consider the maximal control-invariant set $\mathcal{X}_0 \subseteq \mathcal{X}$ given by

$$\mathcal{X}_0 = \{x_0 \in \mathcal{X} \mid \exists\, u(\cdot) \in \mathcal{L}([0, \infty), \mathcal{U}) : \\ \forall t \geq 0\; x(t, x_0, u(\cdot)) \in \mathcal{X}\}, \tag{2}$$

where $\mathcal{L}([0, \infty), \mathcal{U})$ denotes the class of measurable functions on $[0, \infty)$ taking values in the compact set $\mathcal{U} \subset \mathbb{R}^{n_u}$. This set is the largest subset of $\mathcal{X}$ that can be made positively invariant via a control $u(\cdot)$. Here, we assume that $\mathcal{X}_0 \neq \emptyset$. Furthermore, consider a finite-horizon OCP that aims at minimizing the objective functional

$$J_T(x_0, u(\cdot)) = \frac{1}{T} \int_0^T F(x(t), u(t))\, dt, \tag{3}$$

where $F : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is the cost function, and T is the optimization horizon. We assume that F is Lipschitz on $\mathcal{X} \times \mathcal{U}$. The OCP reads

$$\inf_{u(\cdot) \in \mathcal{L}([0,T],\mathcal{U})} J_T(x_0, u(\cdot)) \tag{4a}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{4b}$$

$$\forall t \in [0, T] : \quad x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}. \tag{4c}$$

The pair $(x(\cdot, x_0, u(\cdot)), u(\cdot))$ is called admissible if $u(\cdot) \in \mathcal{L}([0, T], \mathcal{U})$ and if there exists a corresponding absolutely continuous solution $x(\cdot, x_0, u(\cdot))$, which satisfies $x(t, x_0, u(\cdot)) \in \mathcal{X}$ for all $t \in [0, T]$. An optimal solution to (4) is denoted by $u^\star(\cdot)$ and the corresponding state trajectory is written as $x^\star(\cdot, x_0, u^\star(\cdot))$.[2]

Notational remarks: We denote the dependence of optimal solutions to (4) on the initial condition x_0 and the horizon length T by writing $\mathcal{OCP}_T(x_0)$. Whenever it is convenient, input–state pairs are written as $z = (x, u)^T$ and the combined input–state constraints are written as $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$. Throughout this paper, we use the superscript $\bar{\cdot}$ to denote a variable at steady state. Hence, we have $f(\bar{z}) = f(\bar{x}, \bar{u}) = 0$. The set of admissible steady–state pairs is denoted as

$$\bar{\mathcal{Z}} := \{\bar{z} \in \mathcal{Z} \mid 0 = f(\bar{z})\}.$$

Admissible trajectory pairs of $\mathcal{OCP}_T(x_0)$ are abbreviated by $z(\cdot, x_0) = (x(\cdot, x_0, u(\cdot)), u(\cdot))^T$. For any function φ with domain $\mathbb{R}^{n_x+n_u}$ we write

$$\varphi\left(z^\star(t, x_0)\right) := \varphi\left(x^\star(t, x_0, u^\star(\cdot, x_0)),\; u^\star(t, x_0)\right).$$

While $\mathcal{OCP}_T(x_0)$ aims at optimizing the transient performance of system (1), one can as well ask for the best stationary operating conditions. These conditions are given by the following steady–state problem:

$$\inf_{\bar{z} \in \mathbb{R}^{n_x+n_u}} F(\bar{z}) \qquad \text{subject to } \bar{z} \in \bar{\mathcal{Z}} \tag{5}$$

where F is the same as in (3). A globally optimal solution to this static optimization problem is denoted as $\bar{z}^\star$. The set of optimal steady–state pairs is denoted by $\bar{\mathcal{Z}}^\star$, i.e.,

$$\bar{\mathcal{Z}}^\star = \left\{\bar{z}^\star \in \bar{\mathcal{Z}} \mid \bar{z}^\star \text{ is optimal in (5)}\right\}. \tag{6}$$

Henceforth, we assume that $\bar{\mathcal{Z}}^\star \neq \emptyset$. The sets $\bar{\mathcal{X}}$ and $\bar{\mathcal{X}}^\star$, with $\bar{\mathcal{X}}^\star \subseteq \bar{\mathcal{X}} \subset \mathcal{X}$, denote the projection of $\bar{\mathcal{Z}} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ onto the state space $\mathbb{R}^{n_x}$ and the projection of $\bar{\mathcal{Z}}^\star \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ onto $\mathbb{R}^{n_x}$, respectively.

In the operation of dynamic processes, it is of major interest to know whether the best infinite-horizon performance can be achieved at the best steady state or via permanent transient operation. Optimal operation over an infinite horizon is defined similar to (Angeli, Amrit & Rawlings, 2012) and (Grüne, 2013) as follows.

Definition 1 (*Optimal Operation at Steady State*)**.** System (1) is said to be optimally operated at steady state if there exists a $\bar{z} = (\bar{x}, \bar{u})^T \in \bar{\mathcal{Z}}$ such that, for any initial condition $x_0 \in \mathcal{X}_0$ and any infinite-time admissible pair $z(\cdot, x_0)$, we have

$$\liminf_{T \to \infty} J_T(x_0, u(\cdot)) \geq F(\bar{z}). \tag{7}$$

The following lemma follows trivially from the above.

Lemma 1. *If system (1) is optimally operated at $\bar{z}$, then $\bar{z}$ is an optimal solution to (5).*

1.2. Turnpike properties of OCPs

Since there is no generally valid definition of turnpike properties of continuous-time OCPs, we propose a definition motivated by a turnpike result given by Carlson et al. (1991). To this end, consider the placeholder variable $\xi \in \{x, z\}$, which, depending

[2] Here, we assume for simplicity that the optimal solution exists and is attained. We refer to Lee and Markus (1967) and Vinter (2010) for conditions ensuring the existence of optimal solutions to OCP (4).