Automatica 81 (2017) 369–376

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Exponential stability of nonlinear differential repetitive processes with applications to iterative learning control[☆]

Berk Altın[a], Kira Barton[b]

[a] Department of Computer Engineering, University of California, Santa Cruz, CA 95064, United States

[b] Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109, United States

ARTICLE INFO

Article history:
Received 27 November 2015
Received in revised form
22 January 2017
Accepted 28 February 2017

Keywords:
Recursive control algorithms
Lyapunov stability
Nonlinear systems
Learning control
Iterative methods

ABSTRACT

This paper studies exponential stability properties of a class of two-dimensional (2D) systems called differential repetitive processes (DRPs). Since a distinguishing feature of DRPs is that the problem domain is bounded in the "time" direction, the notion of stability to be evaluated does not require the nonlinear system defining a DRP to be stable in the typical sense. In particular, we study a notion of exponential stability along the discrete iteration dimension of the 2D dynamics, which requires the boundary data for the differential pass dynamics to converge to zero as the iterations evolve. Our main contribution is to show, under standard regularity assumptions, that exponential stability of a DRP is equivalent to that of its linearized dynamics. In turn, exponential stability of this linearization can be readily verified by a spectral radius condition. The application of this result to iterative learning control (ILC) is discussed. Theoretical findings are supported by a numerical simulation of an ILC algorithm.

1. Introduction

For recursive nonlinear systems in the explicit form

$$\begin{cases} \dot{x}_{k+1}(t) = f(x_{k+1}(t), y_k(t), t), \\ y_{k+1}(t) = g(x_{k+1}(t), y_k(t), t), \end{cases} \tag{1}$$

where $(t, k) \in [0, T] \times \{0, 1, \dots\}$ for some $T \in [0, \infty)$, we are interested in finding necessary and sufficient conditions that establish local exponential stability. The vectors $x_k(t) \in \mathbb{R}^n$ and $y_k(t) \in \mathbb{R}^m$ of this model represent the state and output, respectively. To uniquely determine the solution of (1), it will be necessary to specify boundary conditions y_0 and $\mathbf{x}(0) \triangleq \{x_{k+1}(0)\}_{k=0}^{\infty}$.

Roughly speaking, the notions of stability to be studied throughout this paper will be weak, in the sense that they will not require the one-dimensional (1D) control system given by f to be stable. For example, exponential stability of (1) will imply that the function sequence $\{y_k\}_{k=0}^{\infty}$ converges exponentially to zero in an appropriate signal norm, provided the boundaries are small, and $\mathbf{x}(0)$ also converges exponentially to zero. The precise meaning of stability for this class of systems will be defined later in Section 2.

The nonlinear system (1) appears in many practical problems of interest and falls into the larger class of two-dimensional (2D) dynamic systems called repetitive (or multipass, earlier in the literature) processes (not to be confused with repetitive control), in which information propagation occurs along two axes of independent variables. These processes are characterized by a sequence of passes with *finite length* that act as forcing functions on the dynamics of future passes (Rogers, Eric, Galkowski, & Owens, 2007): The output solution sequence $\{y_k\}_{k=0}^{\infty}$ of (1) can be found by applying the nonlinear system with differential dynamics described by the functions f and g in a repetitive manner. Hence, we will call any system of the form (1) a *differential repetitive process (DRP)*. The counterpart of the DRP (1) in the broader 2D systems theory, where it is assumed that $T = \infty$, will be called a 2D mixed continuous–discrete time system.

The repetitive process paradigm arises in the modeling of certain engineering applications such as long wall coal cutting (Edwards, 1974) and metal rolling (Foda & Agathoklis, 1992). A rich set of examples to these systems can also be found on a more abstract level since recursive algorithms for 1D dynamic systems can be treated as repetitive processes; e.g. iterative solutions to nonlinear optimal control problems (Gupta, Hudson, Bloch, & Kolmanovsky, 2013), nonlinear inversion methods (Devasia, Chen,

☆ This work was supported by the NSF grant CMMI-1334204, and conducted while the first author was with the Department of Electrical Engineering and Computer Science at the University of Michigan. This paper was recommended for publication in revised form by Associate Editor Michael Cantoni under the direction of Editor Richard Middleton.

E-mail addresses: berkaltin@ucsc.edu (B. Altın), bartonkl@umich.edu (K. Barton).

http://dx.doi.org/10.1016/j.automatica.2017.04.004
0005-1098/

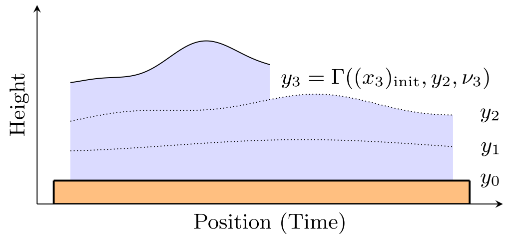

Fig. 1. AM systems as repetitive processes: The substrate topography determines the initial output y_0. The operator Γ maps the initial state $(x_3)_{\mathrm{init}}$ and input v_3 of pass 3 (in-layer dynamics), along with the prior pass profile y_2 (layer-to-layer dynamics), to pass profile y_3. The layer-to-layer dynamics is affected by physical phenomena such as material curing.

& Paden, 1996), iterative estimation and control design (Albertos & Sala, 2002), or the constructive proof of the Picard–Lindelöf theorem. A well-known class of algorithms that can be expressed in the repetitive process framework is iterative learning control (ILC) (Ahn, Chen, & Moore, 2007; Hladowski et al., 2010; Kurek & Zaremba, 1993), wherein the inverse image of a desired output under a 1D input–output system is constructed through a recurrence relation inducing pass-to-pass dynamics. This problem will be tackled in Section 5.

The study of DRPs and other 2D systems bearing similarities with (1) has a long history, beginning with the Roesser and Fornasini–Marchesini models introduced in the 1970s (Fornasini & Marchesini, 1976, 1978; Roesser, 1975). In particular, stability and performance properties of DRPs and 2D mixed continuous–discrete time systems, along with corresponding control strategies, have been researched extensively, *predominantly for linear time-invariant (LTI) systems*—see Chesi and Middleton (2014) and Rogers et al. (2007) and references therein. On the other hand, the need to develop rigorous stability tests in the nonlinear systems context has been highlighted only very recently. Among these works, Yeganefar, Yeganefar, Ghamgui, and Moulay (2013) present forward and converse Lyapunov theorems for nonlinear Roesser models, with extensions to the stochastic case given in Pakshin, Galkowski, and Rogers (2011), and a 2D Lyapunov function approach is employed to prove exponential stability of DRPs in Emelianov, Pakshin, Galkowski, and Rogers (2014). It is also worth noting that the DRP (1) can be viewed as an infinite-dimensional hybrid system (Barreiro & Baos, 2010; Liu & Teel, 2016) by concatenating the passes; e.g. by letting $x(\tau, k+1) \triangleq x_{k+1}(t)$ with $\tau = t + kT$, subject to the periodic reset $x(kT, k+1) = x_{k+1}(0)$, where T plays the role of an inherent delay, τ the ordinary time, and k the jump time/index. As this reset function would change based on the prespecified boundary condition $\mathbf{x}(0)$ and lacks any other structure, we will not follow a hybrid systems approach in the ensuing analysis. See also Rogers et al. (2007) for DRP modeling of a class of delay differential equations.

The objective of this paper is to contribute to the recent literature on nonlinear repetitive process and 2D systems literature, and provide a connection between nonlinear DRPs of the form (1) and their linear counterparts. Therefore, our aim is to certify local exponential stability of DRPs via an appropriate linearization of (1), and establish an analogue of the classical result that exponential stability of a 1D system is equivalent to that of its linear approximation, thereby expanding on the findings of Altın and Barton (2015). Our primary motivation for this study comes from additive manufacturing (AM) systems, wherein material in the fluid phase is often deposited in a layer-by-layer fashion (Fig. 1), leading to 2D dynamics: For instance, the laser metal deposition (LMD) process is characterized by 1D (in-layer) dynamics that are height dependent due to heat transfer from prior layers (Sammons, Bristow, & Landers, 2013). It is possible to achieve accurate material distribution for the LMD process via linear repetitive process control techniques and a more control-oriented model consisting of static nonlinearities. This, however, requires the implicit assumption that the controlled nonlinear process is *locally stable around its linearized equilibrium* (Sammons, Bristow, & Landers, 2014). As a secondary motivation, in the ILC literature, it has been noted that nonlinear update laws have not been researched, save for adaptive laws for locally Lipschitz plants, and a systematic theory of nonlinear ILC is an open question (Xu, 2011).

The rest of the paper is organized as follows: Section 2 introduces the necessary background, establishes the key Lipschitz property of the nonlinear operator, and states formal stability definitions. Stability theory for LTI systems is extended to the linear time-varying (LTV) case in Section 3. Our main result, which establishes equivalence in terms of exponential stability between a DRP and its linearization, is presented in Section 4. Applications of this result to exponential stability analysis of ILC are discussed in Section 5. An illustrative example is given in Section 6 through an ILC system. Concluding remarks are given in Section 7. In the hope of improving readability of the paper, the proof of Proposition 14 is given in the Appendix. Proofs of certain immediate technical results are omitted for brevity and can be found in Altın and Barton (2017).

2. Background and preliminaries

This section will introduce the background material pertinent to our analysis, and lay out stability definitions for the DRP (1). The precise definitions of stability to be presented will show the crucial difference between DRPs and 2D mixed continuous–discrete time systems, as the latter studies the trajectory of the real vector $y_k(t)$ over $\{0, 1, \ldots\} \times [0, \infty)$. In linear repetitive process theory, the gap between these two classes of systems is bridged via the stronger notion of *stability along the pass* (Rogers et al., 2007), which requires the stability parameters to be T independent. Although this property is desirable in experimental implementations or numerical simulations, we will forgo this requirement for theoretical purposes.

Notation. We use $\mathbb{R}$ to represent real numbers, $\mathbb{N}$ nonnegative integers, and $\mathbb{C}$ complex numbers. The spectral radius of a linear operator is denoted by $\rho(.)$. The identity and zero operators are denoted as I and 0, respectively. For a real vector, $\|.\|$ denotes any of the equivalent norms in $\mathbb{R}^p$. $\mathcal{L}_p$ is the space of Lebesgue measurable functions on the compact interval $[0, T]$ with finite $\mathcal{L}_p$ norm, $p \in [1, \infty]$. The space of all sequences on $\mathbb{R}^p$ which converge to 0 is denoted as c_0.

The inequalities below, stated without proof, will be of use for convergence analysis. Note that the convergence parameters $2/(1-a) \geq 1$ and $(1+a)/2 \in (0, 1)$ are continuous increasing functions of a on $(0, 1)$.

Claim 1. *Let* $\mathbf{a} \triangleq \{a_{k+1}\}_{k=0}^{\infty}$ *and* $\mathbf{b} \triangleq \{b_{k+1}\}_{k=1}^{\infty}$ *be real nonnegative sequences, where* $\mathbf{b}$ *is bounded. Suppose that* $a_{k+1} = ra_k + b_{k+1}$ *for some* $r \in (0, 1)$ *for all* $k \in \mathbb{N}$. *Then,* $\limsup_{k\to\infty} a_k \leq (1/(1-r)) \limsup_{k\to\infty} b_k$, *and therefore* $\mathbf{b} \in c_0$ *implies* $\mathbf{a} \in c_0$.

Claim 2. *Let* $a \in (0, 1)$. *Then the sequence* $\{ka^{k-1}\}_{k=0}^{\infty}$ *is exponentially convergent and*

$$ka^{k-1} \leq \frac{2}{1-a}\left(\frac{1+a}{2}\right)^k, \quad \forall k \in \mathbb{N}.$$