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Brief paper

Optimal control for pointwise asymptotic stability in a hybrid control system☆

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ABSTRACT

Pointwise asymptotic stability, or semistability, is a property of the set of equilibria of a dynamical system, where every equilibrium is Lyapunov stable and every solution is convergent to some equilibrium. Under an appropriate version of asymptotic controllability assumption, it is shown that the property can be achieved in a hybrid control system by open-loop optimal solutions of an infinite-horizon optimal control problem. For discrete-time systems, the optimal solutions can be generated by feedback. Regularity of the optimal value function and the existence of hybrid optimal controls are also studied.

1. Introduction

The set of equilibria of a dynamical system is *pointwise asymptotically stable* (PAS), also termed *semistable*, if every equilibrium is Lyapunov stable and every solution converges to one of the equilibria. This stability concept is appropriate for dynamical systems that have a continuum of equilibria (Bhat & Bernstein, 2003), in which case no equilibrium can be asymptotically stable in the usual sense, and have seen application to consensus algorithms (Goebel, 2011; Hui, Haddad, & Bhat, 2008), where the continuum of equilibria consists of consensus states; of hysteresis (Oh, Drincic, & Bernstein, 2009); etc. Semistability/PAS has been studied in continuous-time (Bhat & Bernstein, 2003; Hui, Haddad, & Bhat, 2009), discrete-time (Goebel, 2014b; Hui, 2012), and switched, impulsive, and hybrid systems (Goebel & Sanfelice, 2016a,b; Hui, 2010, 2011b).

Optimal control has played a key role in the design of stabilizing feedback for the usual asymptotic stability, from the classical LQR approach to linear systems (Anderson & Moore, 1990), through nonlinear and constrained discrete-time systems (Keerthi & Gilbert, 1988), to nonlinear and constrained continuous-time systems (Clarke & Stern, 2003). Hybrid optimal control has seen increasing interest, see (Garavello & Piccoli, 2005) and the references therein. Hybrid control systems (Sanfelice, 2013) based on the framework of hybrid inclusions of Goebel, Sanfelice, and Teel (2012), which combine differential equations or inclusions, difference equations or inclusions, and constraints on the resulting motions, have found applications, for example, in the analysis of gene networks (Shu & Sanfelice, 2014), in the estimation of states in mechanical systems with impacts (Forni, Teel, & Zaccarian, 2013), and in the design of control for communication channels (Forni, Galeani, Nešić, & Zaccarian, 2014). It is thus a natural question whether optimal control can help in the design of feedback that results in PAS.

Section 4 proposes an infinite-horizon optimal control problem for a hybrid control system with a set of controlled equilibria, so that open-loop optimal solutions render the set of equilibria PAS. The optimal control problem is inspired in part by Bhat and Bernstein (2010), where arc-length-based sufficient Lyapunov conditions for semistability in continuous-time were given. A hybrid version of the result by Bhat and Bernstein (2010), Theorem 2.2, is in Section 3.[1] Accordingly, the optimal control problem penalizes the norm of the control, and the Lyapunov inequalities satisfied by the optimal value function ensure that optimal solutions have finite length. Combined with a penalty on

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[1] The result appears in the conference paper (Goebel & Sanfelice, 2016b) without proof.

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the distance from the set of equilibria, this yields PAS. In comparison, optimal control approaches to semistability in continuous-time systems in Hui (2011a, 2012), or L'Afflitto, Haddad, and Hui (2015) use quadratic costs and require further assumptions of Lyapunov stability of each equilibrium to guarantee semistability. Furthermore, these approaches associate a separate optimal control problem with each equilibrium. In contrast, the problem in Section 4 does not penalize the distance from any particular equilibrium but from the whole set of them.

In discrete time, optimal solutions that result in PAS can be generated by feedback, as shown in Section 6.[2] Related existing feedback designs for PAS, in continuous or discrete-time, appear limited to linear dynamics, for example second integrators in continuous time in Hui et al. (2008), and a similar comment applies to the related consensus literature; see (Qu, Wang, & Hull, 2008) and (Kim, Shim, & Seo, 2011) and the references therein. An approach to achieving PAS by feedback for a nonlinear continuous-time control system, proposed by the author in Goebel (2014a) and based on the patchy feedback idea (Ancona & Bressan, 2003) appears to have a flaw. Given the technical challenges in formulating optimal feedback for a general nonlinear control problem in continuous time (see the survey Frankowska, 2010 and the references therein) or in feedback stabilization of nonlinear systems in continuous time (Clarke & Stern, 2003), optimal feedback for PAS in a continuous-time or hybrid system is left for future research.

Existence of optimal control is far more challenging in continuous time than in discrete time, and has been treaded extensively in the literature. For hybrid control systems in the framework of Goebel et al. (2012) and Sanfelice (2013), there appears to be no general existence results. Without addressing the continuous-time case, for the problem from Section 4, it is shown in Section 5 that if existence and lower semicontinuity of the optimal value function for the continuous-time part of the hybrid system hold, then so they do for the hybrid system, under a mild condition on the data.

2. Hybrid inclusions and pointwise asymptotic stability

Hybrid inclusions, as represented below, model hybrid dynamical systems.

$$\begin{array}{ll} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x). \end{array} \tag{1}$$

For details, see Goebel, Sanfelice, and Teel (2009); Goebel et al. (2012). Above, $C, D \subset \mathbb{R}^n$ are sets, and $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings.

A set $E \subset \mathbb{R}^2$ is a hybrid time domain if it is a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_1 \le t_2 \le \cdots$, with the last interval, if it exists, possibly of the form $[t_j, t_{j+1})$ or $[t_j, \infty)$. A *hybrid arc* is a function $\phi : \mathrm{dom}\phi \to \mathbb{R}^n$, where the domain $\mathrm{dom}\phi$ of ϕ is a hybrid time domain and, if $I_j(\phi) := \{t \mid (t, j) \in \mathrm{dom}\phi\}$ has nonempty interior, then $t \mapsto \phi(t, j)$ is locally absolutely continuous on I_j. A hybrid arc $\phi : \mathrm{dom}\phi \to \mathbb{R}^n$ is a *solution* to (1) if $\phi(0, 0) \in \overline{C} \cup D$, where $\overline{C}$ is the closure of C, and

- if $I_j(\phi)$ has nonempty interior $\mathrm{int}I_j(\phi)$, then $\phi(t, j) \in C$ for all $t \in \mathrm{int}I_j(\phi)$ and $\frac{d}{dt}\phi(t, j) \in F(\phi(t, j))$ for almost all $t \in I_j(\phi)$;
- if $(t, j) \in \mathrm{dom}\phi$ and $(t, j+1) \in \mathrm{dom}\phi$ then $\phi(t, j) \in D$ and $\phi(t, j+1) \in G(\phi(t, j))$.

A solution $\phi : \mathrm{dom}\phi \to \mathbb{R}^n$ is *maximal* if it cannot be extended, and *complete* if $\mathrm{dom}\phi$ is unbounded. Below, $\mathcal{S}$ is the set of all maximal solutions to (1), $\mathcal{S}(x)$ is the set of maximal solutions to (1) that start from x.

Definition 2.1. A set $A \subset \mathbb{R}^n$ is *pointwise asymptotically stable* for (1) if:

(a) every $a \in A$ is *Lyapunov stable*: for every $a \in A$, every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $\phi \in \mathcal{S}$ with $\|\phi(0, 0) - a\| < \delta$ and every $(t, j) \in \mathrm{dom}\phi$, one has $\|\phi(t, j) - a\| < \varepsilon$; and

(b) every $\phi \in \mathcal{S}$ is bounded, and if it is complete, then $\lim_{t+j\to\infty} \phi(t, j)$ exists and belongs to A.

Pointwise asymptotic stability, as defined above, is global, in contrast to the local concept that requires (b) only for solutions from a neighborhood of A. The usual understanding of asymptotic stability of a set A does not require that every point in A be a Lyapunov stable equilibrium, or even just an equilibrium. It also does not require that solutions have limits; rather, it requires that the distance of every solution from A decrease to 0.

Theorem 2.2. *Suppose that there exists a function $V : \mathbb{R}^n \to \mathbb{R}$, positive definite with respect to a nonempty and closed set A, continuous at every $a \in A$, and such that there exist constants $\overline{c}, \overline{d} > 0$ and continuous functions $c, d : \mathbb{R}^n \to [0, \infty)$, positive definite with respect to A, such that the following two conditions hold:*

(a) for every $T > 0$ and every solution $\phi : [0, T] \to \mathbb{R}^n$ to $\dot{x} \in F(x)$ satisfying $\phi(t) \in C$ for $t \in (0, T)$,

$$V(\phi(t)) + \int_0^T c(\phi(t))\, dt + \overline{c} \int_0^T \|\dot{\phi}(t)\|\, dt \le V(\phi(0)),$$

(b) for every $x \in D$ and every $g \in G(x)$,

$$V(g) + d(x) + \overline{d}\|g - x\| \le V(x).$$

Then A is pointwise asymptotically stable for (1).

Proof. Let $\phi \in \mathcal{S}$. The triangle inequality yields, for $(T_1, J_1), (T_2, J_2) \in \mathrm{dom}\phi$ with $T_1 + J_1 \le T_2 + J_2$, that $\|\phi(T_1, J_1) - \phi(T_2, J_2)\|$ is bounded above by

$$\sum_{j=J_1}^{J_2} \int_{t_j}^{t_{j+1}} \|\dot{\phi}(t_j, j)\|\, dt + \sum_{j=J_1+1}^{J_2} \|\phi(t_j, j) - \phi(t_j, j-1)\| \tag{2}$$

where $(t_j, j), j \le J$, and $t_{J_2+1} = T_2$ describe the hybrid time domain $\mathrm{dom}\phi$ between (T_1, J_1) and (T_2, J_2):

$$\mathrm{dom}\phi \cap ([T_1, T_2] \times [J_1, J_2]) = \bigcup_{j=J_1}^{J_2} [t_j, t_{j+1}] \times \{j\}. \tag{3}$$

By (a) and (b), $\alpha\,(V(\phi(J_1, T_1)) - V(\phi(J_2, T_2)))$ bounds (2) from above, where $\alpha = 1/\min\{\overline{c}, \overline{d}\}$, and so

$$\|\phi(T_1, J_1) - \phi(T_2, J_2)\| \le \alpha\,(V(\phi(J_1, T_1)) - V(\phi(J_2, T_2)))\,. \tag{4}$$

In particular, $\|\phi(0, 0) - \phi(t, j)\| \le \alpha V(\phi(0, 0))$ for every $(t, j) \in \mathrm{dom}\phi$ and ϕ is bounded.

Pick $a \in A$ and $\varepsilon > 0$. Pick $\delta \in (0, \varepsilon/2)$ so that $\|x - a\| < \delta$ implies $V(x) < \varepsilon/(2\alpha)$, which is possible since $V(a) = 0$ and V is continuous at a. By (4), every solution ϕ with $\|\phi(0, 0) - a\| < \delta$ satisfies $\|\phi(t, j) - \phi(0, 0)\| < \alpha V(\phi(0, 0)) < \varepsilon/2$. Since $\|\phi(0, 0) - a\| < \varepsilon/2$, $\|\phi(t, j) - a\| < \varepsilon$ for every $(t, j) \in \mathrm{dom}\phi$, and this verifies Lyapunov stability of a.

Another consequence of (a) and (b) is that, for every solution ϕ and $(T, J) \in \mathrm{dom}\phi$, with (t_j, j) as above,

$$\sum_{j=0}^{J} \int_{t_j}^{t_{j+1}} c(\phi(t, j))\, dt + \sum_{j=1}^{J} d(\phi(t_j, j-1)) \le V(\phi(0, 0)).$$

If ϕ is complete, for the quantity on the left above to remain bounded as $T + J \to \infty$, there must exist a sequence of points $(t_k, j_k) \in \mathrm{dom}\phi$ with $t_k + j_k \to \infty$ when $k \to \infty$ such that either $c(\phi(t_k, j_k))$ or $d(\phi(t_k, j_k - 1))$ converges to 0. Since ϕ is bounded, without loss of generality one can say that either $\phi(t_k, j_k)$

[2] The discrete-time case was studied in the conference paper (Goebel, 2016), where under further assumptions robustness of the feedback was addressed.