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Brief paper

Fixed-time observer with simple gains for uncertain systems[☆]

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ABSTRACT

In this article, we consider the problem of fixed-time observer for nonlinear systems, that is a finite-time observer whose settling time can be bounded independently of the initial condition. We consider a large class of nonlinear systems which includes two main classes: linearizable systems up to input–output injection and uniformly observable systems. Furthermore, the effect of noise and uncertainty is analyzed.

1. Introduction

Observability of nonlinear systems has been characterized within either a differential geometric framework (Hermann & Krener, 1977) or a differential algebraic framework (Diop & Fliess, 1991a,b). Once this question is answered for a given system then a practical solution to reconstruct the non measured state has to be worked out. This motivated an increasing number of works on nonlinear observers design during the last decades leading to a great variety of solutions. A first one, which consists in finding a change of coordinates such that the resulting system is linear, has been considered in Conte, Moog, and Perdon (2007) using algebraic tools. A second one, based on the Lyapunov auxiliary theorem and a direct change of coordinates, can be found in Andrieu and Praly (2006). A third strategy for the class of uniformly observable systems uses high-gain observers (Ahmed-Ali, Cherrier, & Lamnabhi-Lagarrigue, 2012; Esfandiari & Khalil, 1992; Gauthier, Hammouri, & Othman, 1992). Other methods usually rely on a specific structure such as backstepping, adaptive observers, $\mathcal{H}_\infty$ observer, etc.

All the above mentioned approaches deal with asymptotic convergence, while finite-time stability has recently become an active area of research (Bhat & Bernstein, 2000, 2005) for the following reasons: better disturbance rejection and robustness properties (Venkataraman & Gulati, 1991) are obtained (because, for system with finite settling-time, the static equivalent gain is "infinite"), this finite-time response property is a possible tool for separation principle of nonlinear systems and is well adapt to very severe time response constraint. For example, such property is useful for the synchronization of chaotic signals (with application in secure communication) (Perruquetti, Floquet, & Moulay, 2008) or for walking robots (Plestan, Grizzle, Westervelt, & Abba, 2003). Sliding mode observers, which are widely used, allow finite-time convergence, see Floquet and Barbot (2008) and Orlov (2004), but they are not smooth. Other approaches have been considered in order to obtain finite-time convergence such as moving horizon observers in Michalska and Mayne (1995) or delay systems in Menold, Findeisen, and Allgöwer (2003) but they can hardly be generalized to wider classes of nonlinear systems. Quite recently, homogeneity (Battilotti, 2014; Bhat & Bernstein, 2005) was used to obtain finite-time convergence property (Bhat & Bernstein, 2005). Finite-time observers based on recursive construction can be found in Andrieu, Praly, and Astolfi (2008, 2009) but the complexity increases with the system's dimension since the corrective term is composed of nested polynomial terms. Another alternative is to design finite-time observers using as simple gains as the one obtained in the linear case. This track has accomplished considerable progress (Ménard, Moulay, & Perruquetti, 2010; Perruquetti et al., 2008; Shen & Xia, 2008). Other observers with time varying gains or additional assumptions have been developed in Li, Shen, and Xia (2013), Li, Xia, and Shen (2013), Shen and Huang (2009) and Shen and Xia (2010). In this article, we are interested in

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a particular kind of stability, namely fixed-time stability. A system is fixed-time stable if it is finite-time stable and if its settling time can be bounded independently of the initial condition. This terminology has been adopted recently in Polyakov (2012) and Polyakov, Efimov, and Perruquetti (2015).

Here we provide fixed-time convergent observers for a class of nonlinear systems which includes: the class of linearizable systems up to input–output injection and the class of uniformly observable systems, subject to uncertainty and noise. One of the main advantages of the proposed observer is the simplicity of the gains selection, since they are set off-line by solving the same Riccati equation as in the linear case. Furthermore, a general class of corrective terms for the fixed-time convergence is here proposed, generalizing the existing ones (for finite-time convergence). If there is no uncertainty and no noise, the error is proved to converge in finite-time to the origin, whereas in the presence of uncertainty and/or noise, the error converges toward a ball whose radius depends on the bound of the noise and/or uncertainty. In both cases, the settling time can be bounded independently of the initial conditions. The two last features are new, indeed, to the authors' best knowledge the simplest gains proposed in the literature are those given in Ménard et al. (2010) and Shen and Xia (2008), the first one is only semi-global and the second one does not provide a bound independent of the initial conditions for the settling time. Furthermore the effect of uncertainty and noise has never been studied for this approach.

The article is organized as follows. Section 2, provides some notations and definitions. In Section 3 a fixed-time observer is proposed for a large class of nonlinear systems. Then, Section 4 gives convergence results in both cases with or without noise/uncertainty. Convincing simulations are given in Section 5. Finally, Section 6 concludes the article.

2. Notations and definitions

In the paper the following notations are used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$;
- $\mathbb{R}_+^n = (\mathbb{R}_+)^n$, with $n \in \mathbb{N}$;
- $\lceil x \rfloor^\alpha = \mathrm{sign}(x) \cdot |x|^\alpha$, with $\alpha > 0$ and $x \in \mathbb{R}$;
- $\| \cdot \|$ denotes the euclidean norm;
- $\lambda_m(M)$ and $\lambda_M(M)$ are respectively the lowest and the greatest eigenvalue of the square matrix M;
- $\delta_\lambda^r \triangleq \mathrm{diag}(\lambda^{r_1}, \dots, \lambda^{r_n})$ for all $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ and $\lambda \in \mathbb{R}_+$;
- the function ν is defined as

$$\nu(x, \alpha, \beta) = \begin{cases} \lceil x \rfloor^\alpha & \text{if } |x| < 1 \\ \lceil x \rfloor^\beta & \text{if } |x| \geq 1, \end{cases} \tag{1}$$

  with $x \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$;
- the matrix $A \in \mathbb{R}^{n \times n}$ and the vectors $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ are defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T; \tag{2}$$

- $\Delta_\theta \triangleq \mathrm{diag}\left(1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}\right)$ with $\theta \geq 1$;
- $F(K, x, \alpha) \triangleq (k_1 \lceil x \rfloor^{\alpha_1}, \dots, k_n \lceil x \rfloor^{\alpha_n})^T$ with $x \in \mathbb{R}$, $\alpha > 0$ and $K = (k_1, \dots, k_n)$;
- a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}_\infty$ if ϕ is strictly increasing, $\phi(0) = 0$ and $\lim_{r \to +\infty} \phi(r) = +\infty$.

One of the key-properties used in this article is homogeneity which is defined hereafter.

Definition 1. A function $V : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree d* with respect to the weights $(r_1, \dots, r_n) \in \mathbb{R}_+^n$ if $V(\delta_\lambda^r x) = \lambda^d V(x)$, for all $\lambda > 0$ and $x \in \mathbb{R}^n$. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is *homogeneous of degree d* with respect to the weights $(r_1, \dots, r_n) \in \mathbb{R}_+^n$ if for all $1 \leq i \leq n$, the ith component f_i is a homogeneous function of degree $r_i + d$. A dynamical system $\dot{x} = f(x)$ is homogeneous of degree d if the vector field f is homogeneous of degree d.

Let $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field, such that $f(0, 0) = 0$. Consider the system

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(0) = x_0. \end{cases} \tag{3}$$

Definition 2 (*Polyakov et al., 2015*)**.**

- The equilibrium point $x = 0$ of the system (3) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t)$ starting from x_0 reaches the equilibrium at some finite moment, i.e. $x(t) = 0$, for all $t \geq T(x_0)$, where $T : \mathbb{R}^n \to \mathbb{R}^+$ is the so-called settling time function.
- If in addition the settling-time function $T(x_0)$ is bounded by some positive number $T_{\max} > 0$, i.e. $T(x_0) \leq T_{\max}$, for all $x_0 \in \mathbb{R}^n$, the system (3) is said to be globally fixed-time stable.

Remark 1. This last definition can be easily adapted for a compact set instead of an equilibrium point.

3. Fixed-time observer design

When designing observers for nonlinear systems, two main classes of systems are considered: linearizable systems up to input–output injections and uniformly observable systems. In the first case, methods for turning systems into a linear system up to input–output injection through diffeomorphism can be found in Califano and Moog (2014), Krener and Isidori (1983), Krener and Respondek (1985), and through immersion in Back, Yu, and Seo (2006) and Jouan (2003). In the second case of uniformly observable systems, due to the nonlinearity structure of the obtained system, authors use mainly high-gain linear corrective term leading to at least a semi-global exponential convergence of the error (global convergence can be performed when the nonlinearity has a Lipschitz property). Due to the obtained structure different corrective terms can be selected:

- proportional to the output error, which gives exponential convergence,
- proportional to the power of the output error (with power less than one) as in Perruquetti et al. (2008) and Shen and Xia (2008), which gives semi-global finite-time convergence,
- a linear combination of the two previous ones, which gives global finite-time convergence.

Here, we will design an observer whose observation error converges to zero in fixed time: for this, a new nonlinear corrective term will be introduced.

3.1. The class of system under consideration

We consider the class of uncertain systems which are diffeomorphic to the following triangular form up to input–output