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Brief paper

Rank-constrained optimization and its applications[☆]

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ABSTRACT

This paper investigates an iterative approach to solve the general rank-constrained optimization problems (RCOPs) defined to optimize a convex objective function subject to a set of convex constraints and rank constraints on unknown rectangular matrices. In addition, rank minimization problems (RMPs) are introduced and equivalently transformed into RCOPs by introducing a quadratic matrix equality constraint. The rank function is discontinuous and nonconvex, thus the general RCOPs are classified as NP-hard in most of the cases. An iterative rank minimization (IRM) method, with convex formulation at each iteration, is proposed to gradually approach the constrained rank. The proposed IRM method aims at solving RCOPs with rank inequalities constrained by upper or lower bounds, as well as rank equality constraints. Proof of the convergence to a local minimizer with at least a sublinear convergence rate is provided. Four representative applications of RCOPs and RMPs, including system identification, output feedback stabilization, and structured H_2 controller design problems, are presented with comparative simulation results to verify the feasibility and improved performance of the proposed IRM method.

1. Introduction

Rank-constrained optimization problems (RCOPs) are to minimize a convex function subject to a convex set of constraints and rank constraints on unknown matrices. They have received extensive attention due to their wide applications in signal processing, model reduction, and system identification, just to name a few (Kim & Moon, 2006; Markovsky, 2008, 2011; Ten Berge & Kiers, 1991). Although some special RCOPs can be solved analytically (Golub, Hoffman, & Stewart, 1987; Markovsky & Van Huffel, 2007), they are NP-hard in most of the cases. Existing methods for RCOPs mainly focus on alternating projection based methods (Dattorro, 2015; Delgado, Agüero, & Goodwin, 2014; Grigoriadis & Skelton, 1996) and combined linearization and factorization algorithms (Hassibi, How, & Boyd, 1999; Meyer, 2011) with application to factor analysis, etc. However, these iterative approaches depend on the initial guess and fast convergence cannot be guaranteed. In addition, a Newton-like method (Orsi, Helmke, & Moore, 2006) has been proposed to search for a feasible solution of RCOPs with application to a feedback stabilization problem. A Riemannian manifold optimization method (Vandereycken, 2010) has been applied to solve large-scale Lyapunov matrix equations by finding a low-rank approximation. Also, Urrutia, Delgado, and Agüero (2016) propose to use the toolbox BARON to solve a RCOP. There are alternative approaches for solving specially formulated RCOPs. For example, a greedy convex smoothing algorithm has been designed to optimize a convex objective function subject to only one rank constraint (Shalev-Shwartz, Gonen, & Shamir, 2011).

When the rank function in constraints of RCOPs appears as the objective of an optimization problem, it turns to be a rank minimization problem (RMP), classified as a category of nonconvex optimization. Applications of RMPs have been found in a variety of areas, such as matrix completion (Candès & Recht, 2009; Mohan & Fazel, 2012; Recht, Fazel, & Parrilo, 2010), control system analysis and design (El Ghaoui & Gahinet, 1993; Fazel, Hindi, & Boyd, 2001, 2004; Mesbahi, 1998; Mesbahi & Papavassilopoulos, 1997), and machine learning (Meka, Jain, Caramanis, & Dhillon, 2008; Recht, 2011). The wide application of RMPs attracts extensive studies aiming at developing efficient optimization algorithms.

Due to the discontinuous and nonconvex nature of the rank function, most of the existing methods solve relaxed or simplified RMPs by introducing an approximate function, such as log-det or nuclear norm heuristic methods (Fazel, 2002; Fazel, Hindi, & Boyd, 2003). The heuristic methods minimize a relaxed convex function instead of the exact rank function over a convex set, which is computationally favorable. They generally generate a solution with lower rank, even a minimum rank solution in special

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cases (Recht et al., 2010). The relaxed formulation with convex objective and constraints does not require the initial guess and global optimality is guaranteed for the relaxed formulation. When the unknown matrix is constrained to be positive semidefinite, relaxation of RMPs using a trace function is equivalent to the relaxed formulation using a nuclear norm function based on the fact that the trace of a positive semidefinite matrix is equal to its nuclear norm (Mesbahi & Papavassilopoulos, 1997). For cases when the unknown matrix is not positive semidefinite, work in Fazel (2002) introduces a semidefinite embedding lemma to extend the trace heuristic method to general cases.

However, a relaxed function cannot represent the exact rank function and performance of the heuristic method is not guaranteed. Other heuristic methods, e.g., the iterative reweighted least square algorithm (Mohan & Fazel, 2012) which iteratively minimizes the reweighted Frobenius norm of the matrix, cannot guarantee the minimum rank solution either. The uncertainty of the performances in heuristic methods stems from the fact that these methods are minimizing a relaxed function and generally there is a gap between the relaxed objective and the exact one. Other methods for RMPs include the alternating projections and its variations (Grigoriadis & Skelton, 1996; Lee & Bresler, 2010; Orsi et al., 2006), linearization (El Ghaoui, Oustry, & AitRami, 1997; Hassibi et al., 1999), and augmented Lagrangian method (Fares, Apkarian, & Noll, 2001). These methods, similar to existing iterative methods for RCOPs, depend on initial guess, which generally leads to slow convergence to just a feasible solution.

After reviewing the literature, we come to a conclusion that more efficient approaches that are applicable for general ROCPs/RMPs with advantages in terms of convergence rate, robustness to initial guess, and performance of cost function, are required to solve RCOPs and RMPs. To our knowledge, there is few literature that addresses equivalent conversion from RMPs into RCOPs (Delgado, Agüero, & Goodwin, 2016; Sun & Dai, 2015a). This paper describes a novel representation of RMPs in the form of RCOPs and proposes a uniform approach to both RCOPs and reformulated RMPs. Therefore, instead of solving two classes of nonconvex optimization problems separately, the uniform formulation and approach significantly reduces the required efforts for solving two types of challenging problems.

An iterative rank minimization (IRM) method, with each sequential problem formulated as a convex optimization problem, is proposed to solve RCOPs. The IRM method was introduced in our previous work to solve quadratically constrained quadratic programming problems which are equivalent to rank-one constrained optimization problems (Dai & Sun, 2015; Sun & Dai, 2015b). The IRM method proposed in this paper aims to solve general RCOPs, where the constrained rank could be any assigned integer number. Although IRM is primarily designed for RCOPs with rank constraints on positive semidefinite matrices, a semidefinite embedding lemma (Fazel, 2002) is introduced to extend IRM to RCOPs with rank constraints on general rectangular matrices. Moreover, the proposed IRM method is applicable to RCOPs with rank inequalities constrained by upper or lower bounds, as well as rank equality constraints. Sublinear convergence of IRM is proved via the duality theory and the Karush–Kuhn–Tucker conditions. To verify the effectiveness and improved performance of proposed IRM method, four representative applications, including system identification, output feedback stabilization, and structured H_2 controller design problems, are presented with comparative results.

The rest of the paper is organized as follows. In Section 2, the problem formulation of RCOP and the conversion of RMP to RCOP are described, including extension to rank constraints on general rectangular matrices. The IRM approach and its local convergence proof are addressed in Section 3. Four application examples and their comparative results are presented in Section 4. We conclude the paper with a few remarks in Section 5.

1.1. Preliminaries

Some notations used throughout this paper are introduced in this section. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$ and the set of $n \times n$ positive semidefinite (definite) matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$). The notation $X \succeq \mathbf{0}$ ($X \succ \mathbf{0}$) means that the matrix $X \in \mathbb{S}^n$ is positive semidefinite (definite). The symbol '$\Leftrightarrow$' means if and only if logical connective between statements. The trace of X is denoted by $\mathbf{trace}(X)$ and the rank of X is denoted by $\mathbf{rank}(X)$. The linear span of vectors in X is denoted by $\mathbf{span}(X)$.

2. Problem formulation

2.1. Rank-constrained optimization problems

A general RCOP to optimize a convex objective subject to a set of convex constraints and rank constraints can be formulated as follows

$$\begin{aligned} \min_{X} \quad & f(X) \\ s.t. \quad & X \in \mathcal{C}, \quad \mathbf{rank}(X) \lesseqqgtr r, \end{aligned} \tag{2.1}$$

where $f(X)$ is a convex function, $\mathcal{C}$ is a convex set, and $X \in \mathbb{R}^{m\times n}$ is a general rectangular matrices set. Without loss of generality, it is assumed that $m \leq n$. The sign $\lesseqqgtr$ includes all types of rank constraints, including upper and lower bounded rank inequality constraints and rank equality constraints. Although lower bounded rank inequality constraints and rank equality constraints do not have as many practical applications compared to the upper bounded rank inequality constraints, they are included here for completeness. Because the existing and proposed approaches for RCOPs require the to-be-determined matrix to be a positive semidefinite matrix, it is then necessary to convert the rank constraints on rectangular matrices into corresponding ones on positive semidefinite matrices.

Lemma 1 (*Lemma 1 in Fazel et al., 2004*). *Let $X \in \mathbb{R}^{m\times n}$ be a given matrix, then $\mathbf{rank}(X) \leq r$ if and only if there exist matrices $Y = Y^T \in \mathbb{R}^{m\times m}$ and $Z = Z^T \in \mathbb{R}^{n\times n}$ such that*

$$\mathbf{rank}(Y) + \mathbf{rank}(Z) \leq 2r, \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \succeq \mathbf{0}.$$

However, Lemma 1 is not applicable to lower bounded rank inequality constraints. As a result, a new lemma is introduced to extend the above semidefinite embedding lemma to all types of rank constraints. Before that, we first describe a proposition which is involved in proof of the new lemma.

Proposition 2. *$Z = X^TX$ is equivalent to $\mathbf{rank}\left(\begin{bmatrix} I_m & X \\ X^T & Z \end{bmatrix}\right) \leq m$, where $Z \in \mathbb{S}^n$, $X \in \mathbb{R}^{m\times n}$, and $I_m \in \mathbb{R}^{m\times m}$ is an identity matrix.*

Proof. Given that the rank of a symmetric block matrix is equal to the rank of a diagonal block plus the rank of its Schur complement, we have the following relationship, $\mathbf{rank}\left(\begin{bmatrix} I_m & X \\ X^T & Z \end{bmatrix}\right) \leq m \Leftrightarrow \mathbf{rank}(I_m) + \mathbf{rank}(Z - X^TX) \leq m \Leftrightarrow m + \mathbf{rank}(Z - X^TX) \leq m \Leftrightarrow \mathbf{rank}(Z - X^TX) = 0 \Leftrightarrow Z = X^TX$. □

Remark. When $Z = X^TX$, it indicates that $\begin{bmatrix} I_m & X \\ X^T & Z \end{bmatrix} \succeq \mathbf{0}$ holds for $I_m \succ \mathbf{0}$ and its Schur complement is a zero matrix.

Next, we give the extended semidefinite embedding lemma below.