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Brief paper

Stability of uncertain systems using Lyapunov functions with non-monotonic terms[☆]

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ABSTRACT

This paper is concerned with the problem of robust stability of uncertain linear time-invariant systems in polytopic domains. The main contribution is to present a systematic procedure to check the stability of the uncertain systems by using an arbitrary number of quadratic functions within higher order derivatives of the vector field in the continuous-time case and higher order differences of the vector field in the discrete-time case. The matrices of the Lyapunov function appear decoupled from the dynamic matrix of the system in the conditions. This fact leads to sufficient conditions that are given in terms of Linear Matrix Inequalities defined at the vertices of the polytope. The proposed method does not impose sign condition constraints in the quadratic functions that compose the Lyapunov function individually. Moreover, some of the quadratic functions do not decrease monotonically along trajectories. However, if the sufficient conditions are satisfied, then a monotonic standard Lyapunov function that depends on the dynamics of the uncertain system can be constructed *a posteriori*. Numerical examples from the literature are provided to illustrate the proposed approach.

1. Introduction

Lyapunov theory has proven to be a powerful tool to guarantee the stability of dynamical systems (Khalil, 2002). Most of the results for stability analysis presented in the literature search for a standard Lyapunov function, i.e., one that must be positive definite and must decrease monotonically along trajectories. Some few works have raised the question of why should we require the Lyapunov function to decrease monotonically. In Butz (1969), the problem of inferring asymptotic stability for continuous-time systems has been addressed without requiring the first derivative of the Lyapunov function to be negative definite. Instead of that, a condition based on the existence of a three times continuously differentiable Lyapunov function has been proposed. The work was extended in Meigoli and Nikravesh (2009) to consider higher order derivatives of the Lyapunov function. However, in both cases the use of higher order derivative of the Lyapunov function has led to non-convex conditions, that rely on the search of scalar parameters and the Lyapunov function at the same time. In Ahmadi and Parrilo (2011) it has been shown that once the conditions in Butz (1969) and Meigoli and Nikravesh (2009) are satisfied, then a standard Lyapunov function can be constructed. The Lyapunov function is parameterized by higher order derivatives of the vector field. It was demonstrated that convex conditions based on the existence of this structured Lyapunov function can be obtained to solve the problem. The counterpart of this result for discrete-time, using higher order differences of the vector field, has been presented in Ahmadi and Parrilo (2008). It is also worth mentioning the work in Sassano and Astolfi (2013) that proposes the use of dynamic Lyapunov functions to characterize the stability of linear and nonlinear systems and the recent approach (Chesi, 2015) that does not rely on the use of Lyapunov functions and can provide a certificate of instability for uncertain systems by means of semidefinite programming and determinants of matrices.

The well known quadratic stability condition has been used as the first method to certify the stability of linear time invariant (LTI) uncertain systems in polytopic domains (Barmish, 1985). However, the use of a common Lyapunov matrix to assure the stability for all the uncertain domain can be conservative in some cases. To reduce the conservatism of the conditions, the main

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developments were in the sense of improving the structure of the Lyapunov matrix, considering affine Lyapunov functions (Geromel, de Oliveira, & Hsu, 1998; Leite & Peres, 2003; Peaucelle, Arzelier, Bachelier, & Bernussou, 2000) and polynomially parameter-dependent Lyapunov functions (Chesi, 2008; Chesi, Garulli, Tesi, & Vicino, 2005; Oliveira & Peres, 2006, 2007; Scherer, 2006) to assure the stability of uncertain systems. In another direction, the authors in Lee, Park, and Joo (2011) have developed an approach to compute stability of LTI uncertain systems using higher order derivatives of the Lyapunov function. As in Meigoli and Nikravesh (2009), the conditions proposed in Lee et al. (2011) depend on scalar parameters, which is the main drawback of the method accordingly to the authors. An algorithm to minimize the effects of the scalar search has been employed, but the results can still be conservative.

In Ebihara, Peaucelle, Arzelier, and Hagiwara (2005) higher order time-derivatives of the states (limited to third order derivatives) have been used to construct a Lyapunov function. The work was extended in Peaucelle, Arzelier, Henrion, and Gouaisbaut (2007), considering a Lyapunov function composed by a generic number of higher order time-derivatives of the states, to deal with the problem of topological separation. Another related work is (Ebihara, Peaucelle, & Arzelier, 2015, Chapter 2), that addresses the robust performance analysis of LTI uncertain systems using conditions with the presence of slack variables. In the main results comments will be provided to clarify the relation between the proposed approach and the use of time-derivatives of the states to construct Lyapunov functions. At this point, it is important to remember that the development of efficient stability conditions is the first step towards achieving effective synthesis conditions.

This paper provides a systematic procedure to check the stability of LTI uncertain systems in polytopic domains. The conditions are based on the existence of a Lyapunov function composed by a generic number of quadratic functions. Higher order derivatives (differences) of the vector field in the continuous-time (discrete-time) case are employed. The quadratic functions do not have sign condition constraints individually, and some of them do not decrease monotonically. The proposed method decouples the matrices of the Lyapunov function from the dynamic matrix of the uncertain system, preventing the computation of power of uncertain matrices that show up in the higher order derivatives of the vector field. This fact leads to sufficient conditions that are given in terms of Linear Matrix Inequalities (LMIs) defined at the vertices of the polytope. If the conditions are fulfilled, then a monotonic Lyapunov function that depends on the dynamic matrix of the system can be constructed *a posteriori*. The proposed approach also contains as particular cases some well known conditions from the literature. Numerical experiments show the potential of the technique of requiring a smaller number of scalar decision variables and LMI rows to certify the stability of uncertain systems.

Notation. For two symmetric matrices of same dimensions A and B, $A > B$ means that $A - B$ is positive definite. For matrices or vectors $(^T)$ indicates transpose. Matrix $He(Z) = Z + Z^T$ is used to simplify the developments. In continuous-time case, $V^p(x)$ represents the derivative of order p of the function $V(x)$, while in the discrete-time case $V^p(x)$ represents the function $V(x)$[1] evaluated at the instant p. $A \otimes B$ represents the Kronecker product between A and B. The factorial of d is denoted by $d!$. Identity (null) matrices of dimension $n \times n$ ($n \times m$) are denoted by I_n ($0_{n\times m}$).

[1] For simplicity of notation, the dependence of $V(x)$ on x is omitted in some of the formulations.

2. Background

Consider the dynamical system

$$\delta[x] = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$. The operator $\delta[x]$ denotes the time-derivative for continuous-time systems and the shift operator for discrete-time systems. We are interested in verifying if $f(0) = 0$ is the unique stable equilibrium point of the system, i.e., if the system is globally asymptotically stable (GAS). Before introducing the main problem of this paper, let us state some results presented in the literature that make use of higher order derivatives (differences) of the vector field in the continuous-time (discrete-time) case to infer GAS of (1).

The first result can be seen as a generalization of the conditions proposed in Butz (1969) and Meigoli and Nikravesh (2009) and has been presented in Ahmadi and Parrilo (2011). Instead of search for scalar parameters and just one single function $V(x)$, convex conditions were obtained by using different functions $V_i(x)$, as stated in the next lemma.

Lemma 1 (*Ahmadi & Parrilo, 2011*). *If there exists a radially unbounded function $W(x)$ such that*

$$W(x) = V_{N+1}^N(x) + V_N^{(N-1)}(x) + \cdots + \dot{V}_2(x) + V_1(x)$$
$$W(0) = 0, \qquad W(x) > 0, \qquad \dot{W}(x) < 0, \quad \forall x \neq 0$$

then the origin is a GAS equilibrium point of (1) and $W(x)$ is a standard Lyapunov function.

Note that Lemma 1 does not require sign condition for any individual function $V_i(x)$, $i = 1, \ldots, N+1$. Moreover, $W(x)$ is a standard Lyapunov function that has been parameterized in a very special way using derivatives of the vector field, which show up in the derivatives of the functions $V_i(x)$.

Concerning the discrete-time case, the following condition is of interest to our work

Lemma 2 (*Ahmadi & Parrilo, 2008*). *If there exist continuous functions $V_1, \ldots, V_{N+1} : \mathbb{R}^n \to \mathbb{R}$ such that $\sum_{i=1}^{N+1} iV_i(0) = 0$,*

$$\sum_{i=j}^{N+1} V_i \text{ radially unbounded for } j = 1, \ldots, N+1$$

$$\sum_{i=j}^{N+1} V_i > 0 \quad \forall x \neq 0 \text{ for } j = 1, \ldots, N+1$$

$$(V_{N+1}^{k+N+1} - V_{N+1}^k) + \cdots + (V_1^{k+1} - V_1^k) < 0$$

then the origin is a GAS equilibrium point of (1) and $W^k(x) = \sum_{j=1}^{N+1}\sum_{i=j}^{N+1} V_i^{k+j-1}$ is a standard Lyapunov function.

3. Problem formulation

Consider the following linear time invariant uncertain system

$$\delta[x] = A(\alpha)x \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector. The uncertain matrix $A(\alpha)$ belongs to a polytopic domain parameterized in terms of a time-invariant vector α, being given by

$$A(\alpha) = \sum_{z=1}^{Z} \alpha_z A_z, \quad \alpha \in \Lambda_Z \tag{3}$$