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Brief paper

The robust minimal controllability problem[☆]

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ABSTRACT

In this paper, we address the robust minimal controllability problem, where the goal is, given a linear time-invariant system, to determine a minimal subset of state variables to be actuated to ensure controllability under additional constraints. We study the problem of characterizing the sparsest input matrices that assure controllability, when the autonomous dynamics' matrix is simple when a specified number of inputs fail. We show that this problem is NP-hard, and under the assumption that the dynamics' matrix is simple, we show that it is possible to reduce the problem to a set multi-covering problem. Additionally, under this assumption, we prove that this problem is NP-complete, and polynomial algorithms to approximate the solutions of a set multi-covering problem can be leveraged to obtain close-to-optimal solutions.

1. Introduction

The problem of guaranteeing that a dynamical system can be driven toward the desired state regardless of its initial position is a fundamental question studied in control systems and it is referred to as *controllability*. Several applications, for instance, control processes, multi-agents networks, control of large flexible structures, systems biology and power systems (Egerstedt, 2011; Siljak, 2007; Skogestad, 2004) rely on the notion of controllability to safeguard their proper functioning. Subsequently, it is important to identify which subsets of state variables need to be actuated, or what is the placement of actuators required, to ensure controllability (Olshevsky, 2014; Pequito, Kar, & Aguiar, 2016a; van de Wal & de Jager, 2001).

Moreover, actuators may malfunction over time due to the adverse nature of the environments where the actuators are deployed, e.g. due to the wear and tear of the materials, or due to external (adversarial) influence of an agent aiming to disrupt the proper functioning of the dynamical system. In fact, a classical example of such malicious attack is the Stuxnet malware incident (Langner, 2011), in which the controller's input response to a tempered measured output lead the system away from its normal operating conditions. Thus, the control designer needs to consider such scenarios, while accounting for the actuator placement (Velde & Carignan, 1984). Additionally, as the systems become larger (i.e., the dimension of their state space), we aim to identify a relatively small subset of state variables that ensure the controllability of the system, for instance, due to economic constraints (Olshevsky, 2014). Consequently, in this paper we address the following natural design question:

$\mathcal{Q}_1$: *What is the minimum number of actuated state variables we need to consider to ensure the controllability of a dynamical system if a specific number of actuators failures occur?*

To formally capture $\mathcal{Q}_1$, we introduce and study the *robust minimal controllability problem* (rMCP) that aims to determine the minimum number of state variables that need to be actuated to ensure system's controllability, under the possible failure of a specified number of actuators. This is a generalization of the minimal controllability problem (MCP) (Olshevsky, 2014), which can be obtained as a particular case of the rMCP when no actuator fails. Therefore, the MCP is the first step to understand resilience and robustness properties of dynamical systems since it unveils which variables need to be actuated.

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Finally, it is important to mention that the rMCP can be stated regarding observability, by invoking the duality between controllability and observability in LTI systems (Hespanha, 2009). In particular, Chen, Kar, and Moura (2015), Fawzi, Tabuada, and Diggavi (2014) and Shoukry and Tabuada (2014) provide necessary and sufficient conditions concerning the sensor deployment to ensure that a reliable estimate of the system is recovered. More importantly, those conditions can be achieved by design, when solving the rMCP. Hence, guaranteeing the design of stable observers to proper monitor the state evolution of an LTI system. Furthermore, the results presented in this paper are for discrete-time, but they are readily applicable to continuous-time LTI systems.

Related Work: This paper follows up and subsumes previous literature by considering the deployment of actuators to ensure controllability under possible actuation failures. When no actuators fail, it extends the results available for the MCP, as we overview next. In Nabi-Abdolyousefi and Mesbahi (2013) the controllability of circulant networks is analyzed by exploring the Popov–Belevitch–Hautus eigenvalue criterion, where the eigenvalues are characterized using the Cauchy–Binet formula. The controllability in multi-agents with Laplacian dynamics was initially explored in Tanner (2004). Later, in Egerstedt, Martini, Cao, Camlibel, and Bicchi (2012) and Rahmani, Ji, Mesbahi, and Egerstedt (2009), necessary and sufficient conditions are given in terms of partitions of the Laplacian graph. In Parlangeli and Notarstefano (2012), the controllability is explored for paths and cycles, and later extended by the same authors to the controllability of grid graphs by means of reductions and symmetries of the graph (Notarstefano & Parlangeli, 2013), and considering dynamics that are scaled Laplacians. In Kibangou and Commault (2014) and Zhang, Camlibel, and Cao (2011), the controllability is studied for strongly regular graphs and distance-regular graphs. Recently, new insights on the controllability of Laplacian dynamics are given regarding the uncontrollable subspace, in Aguilar and Gharesifard (2015) and Chapman and Mesbahi (2014). In addition, in Pasqualetti and Zampieri (2014) the controllability of isotropic and anisotropic networks is analyzed.

Furthermore, Aguilar and Gharesifard (2015) conclude by pointing out that further study of non-symmetric dynamics and controllability is required — which we address in the present paper. Therefore, we consider a much less restrictive assumption: A is a *simple* matrix, i.e., all of its eigenvalues are distinct. Moreover, there are several applications where A satisfies this assumption, for instance, all dynamical systems modeled as random networks of the Erdős–Rényi type (Tao & Vu, 2017), as well as several known dynamical systems used as benchmarks in control systems engineering (Ogata, 2001; Siljak, 1991, 2007).

Observe that the MCP problem presents both continuous and discrete optimization properties, captured by the controllability property and the number of non-zero entries, respectively. To avoid the nature of this problem, in Olshevsky (2014), the non-zero entries of the input matrix were randomly generated. In the present paper, we 'decouple' the continuous and discrete optimization properties, and show that by first solving the discrete nature of the problem, it is always possible to deterministically obtain a solution to MCP in a second phase. Besides, the first step reduces the MCP to the set covering problem — well known to be NP-hard. Nonetheless, the set covering problem is one of the most studied NP-hard problems (probably second only to the SAT problem). Subsequently, although the set covering problem is NP-hard, some subclasses of the problem are equipped with sufficient structure that can be leveraged to invoke a polynomial algorithm that approximate the solution with 'almost' optimality guarantees (Brönnimann & Goodrich, 1995). This contrasts with the approach proposed in Olshevsky (2014), where an approximated solution particular to the MCP problem was provided. In addition, we study the rMCP which has not been previously addressed in the literature. Similarly to the MCP, we show that the rMCP can be polynomially reduced to the *set multi-covering problem*, i.e., a set covering problem that allows the same elements to be covered a predefined number of times. Furthermore, extensions of polynomial approximation algorithms are also available with similar optimality guarantees.

Alternatively, when the parameters of the LTI system are not exactly known, and assumed to be independent, structural systems theory (Dion, Commault, & der Woude, 2003) can be used to address the MCP and rMCP while ensuring *structural controllability*, see Liu, Pequito, Kar, Sinopoli, and Aguiar (2015) and Pequito, and Kar et al. (2016), respectively. Notwithstanding, the tools and conditions to ensure structural controllability are quite different from those adopted in this paper, and a solution to the MSCP is not necessarily a solution to the MCP when the dynamics' matrix is simple (Pequito, Ramos, Kar, Aguiar, & Ramos, 2016b). ∘

Main Contributions of the present paper are as follows: (i) we characterize the exact solutions to the MCP; (ii) we show that for a given dynamics' matrix almost all input vectors satisfying a specified structure are solutions to the MCP; (iii) we show that the rMCP is an NP-hard problem; (iv) we characterize the exact solutions to the rMCP; (v) we prove that the decision version of both MCPs are NP-complete; (vi) we provide approximated solutions to the rMCPs and discuss their optimality guarantees; and, finally, in (vii) we discuss the limitations of the proposed methodology. ∘

The remainder of this paper is organized as follows. In Section 2, we formally state the rMCP addressed in this paper. Next, in Section 3, we review some concepts required to prove the main results of this paper. In Section 4, we present the main results of this paper, i.e., we characterize the solutions to the rMCP, its complexity, and a polynomial algorithm that approximates the solutions. Finally, in Section 5, we provide some examples that illustrate the main results of the paper and discuss the limitations of the proposed methodology.

Notation: We denote vectors by small font letters such as v, w, b and its corresponding entries by subscripts. A collection of vectors is denoted by $\{v^j\}_{j\in\mathcal{J}}$, where the superscript indicates an enumeration of the vectors using indices from a set such as $\mathcal{I}, \mathcal{J} \subset \mathbb{N}$. The number of elements of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. We denote by I_n the n-dimensional identity matrix. Given a matrix A, $\sigma(A)$ denotes the set of eigenvalues of A, the *spectrum* of A. Given two matrices $M_1 \in \mathbb{C}^{n\times m_1}$ and $M_2 \in \mathbb{C}^{n\times m_2}$, the matrix $[M_1\ M_2]$ is the $n \times (m_1 + m_2)$ concatenated complex matrix. The structural pattern of a vector/matrix or a *structural vector/matrix* have their entries in $\{0, \star\}$, where $\star$ denotes a non-zero entry, and they are denoted by a vector/matrix with a bar on top of it. We denote by $A^\intercal$ the transpose of A. The function $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denotes the usual inner product in $\mathbb{C}^n$, i.e., $v \cdot w = v^\dagger w$, where $v^\dagger$ denotes the adjoint of v (the conjugate of $v^\intercal$). With some abuse of notation, $\cdot : \{0, \star\}^n \times \{0, \star\}^n \to \{0, \star\}$ also denotes the map where $\bar{v} \cdot \bar{w} \neq 0$, with $\bar{v}, \bar{w} \in \{0, \star\}^n$ if and only if there exists $i \in \{1, \dots, n\}$ such that $\bar{v}_i = \bar{w}_i = \star$. Additionally, $\|v\|_0$ denotes the number of non-zero entries of the vector v in either $\{0, \star\}^n$ or $\mathbb{R}^n$. Given a subspace $\mathcal{H} \subset \mathbb{C}^n$ we denote by $\mathcal{H}^c$ its complement with respect to $\mathbb{C}$, i.e., $\mathcal{H}^c = \mathbb{C}^n \setminus \mathcal{H}$. With abuse of notation, we will use inequalities involving structural vectors as well — for instance, we say $\bar{v} \geq \bar{w}$ for two structural vectors $\bar{v}$ and $\bar{w}$ if and the only if the following two conditions hold: (i) if $\bar{w}_i = 0$, then $\bar{v}_i \in \{0, \star\}$, and (ii) if $\bar{w}_i = \star$ then $\bar{v}_i = \star$.

2. Problems statement

Under the adverse scenarios of failure or malicious temper of the actuators, the dynamics of the system can be modeled by

$$x(k + 1) = Ax(k) + B_{\mathcal{M}\setminus\mathcal{A}}u(k), \quad (1)$$