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Performance optimization over positive l_∞ cones[☆]

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ABSTRACT

In this paper we study linear systems with positivity type of constraints. First, we consider the case where the input to a system is restricted to be in the positive cone of l_∞, denoted by l_∞^+, and seek to characterize the system's induced norm from l_∞^+ to l_∞. We obtain an exact characterization of this norm which is particularly easy to calculate in the case of LTI systems. Furthermore, we consider and solve the model matching problem, and show that time-varying linear or nonlinear control/filtering does not improve the performance with respect to this norm for LTI systems. In the second part of the paper, we consider the case when the output is forced to be in the positive l_∞ cone when the input is in this cone. We show if internal positivity is sought, a dynamic optimal controller offers no advantage over a static one. Also, if the measurement matrix satisfies certain conditions, synthesizing an optimal static output feedback controller reduces to a linear program.

1. Introduction

There are many dynamical systems in which some variables are restricted to be non-negative (or non-positive); examples can be found in biology, economics, and many other areas (Berman & Plemmons, 1979; Ledzewicz, Naghnaeian, & Schättler, 2011, 2012). Motivated by such problems, the theory of positive systems has been the focus of many researchers. Notions such as stability, stabilizability, positive realization, and (distributed) control synthesis of such systems have been the subject of research, see e.g. Farina and Rinaldi (2011), Haddad, Chellaboina, and Hui (2010), Kaczorek (2002), and Kaszkurewicz and Bhaya (2012).

For linear systems, the notion of internal positivity refers to the case when the states of the system remain nonnegative if the inputs and the initial conditions are nonnegative. Many aspects of positive linear systems have been investigated extensively, see for example Fornasini and Valcher (2010). The controllability of linear positive systems is studied in Valcher (1996). The problem of positive realization is considered in Farina (1996) and Van Den Hof (1997). Authors of Shafai, Chen, and Kothandaraman (1997) presented explicit formulas for the stability radii of such systems; and, Shafai, Ghadami, and Oghbaee (2013) address the stabilization problem while maximizing the stability radius. Furthermore, the input–output properties and in particular the gains of such systems have been given major attention in Briat (2013), Ebihara, Peaucelle, and Arzelier (2012), Rantzer (2011), and references therein. In Briat (2013), copositive linear Lyapunov functions and linear supply rates are used, in the context of dissipativity theory, to investigate robust stability and performance. Further, the problem of synthesizing an optimal l_∞-induced static state-feedback controller with given sparsity or boundedness constraints is considered and solved. Synthesizing an optimal l_1-induced static state-feedback controller is studied in Chen, Lam, Li, and Shu (2013) and Ebihara et al. (2012). In the latter, the problem is reduced to a bilinear program and an iterative algorithm is utilized to solve it. The output feedback, however, is a more challenging problem. This problem, in general, can be cast as a bilinear program. In Ait-Rami (2011), a linear program is provided to find a rank one static output-feedback gain such that the closed loop system is stable and internally positive. For l_2 type of performance, one can refer to Rantzer (2011), and Tanaka and Langbort (2010, 2011).

In this paper, we are interested in characterizing and optimizing the l_∞ gain of linear systems that contain positivity type of constraints. Two cases are considered: when the input to the

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system is positive and when the system itself is positive. In the first case, when the input is restricted to be in the positive cone of l_∞, denoted by l_∞^+, we seek to characterize the induced norm from l_∞^+ to l_∞. That is, for a given (not necessarily positive) linear system G, we are interested to find $\sup_t \|(Gu)(t)\|_\infty$, where $0 \le u(k) \le 1$ (the inequalities are taken component wise) for all nonnegative integers k. We obtain an exact characterization of this norm (the induced norm from l_∞^+ to l_∞) in terms of the standard l_∞ induced norms of appropriately defined subsystems which is particularly easy to calculate in the case of Linear Time Invariant (LTI) systems. We emphasize that no positivity assumption is made on the system itself. We further consider the more general asymmetric input signals and characterize the input–output gain of such systems. More precisely, for two real numbers a and b, we compute $\sup_t \|(Gu)(t)\|_\infty$, where $a \le u(k) \le b$ for all nonnegative integers k. As an application of the above developments, we consider a filtering problem in which the signal to be estimated, s, is known to live in a positive cone, i.e. $s \in l_\infty^+$. In general, just designing a filter to minimize the standard l_∞ induced norm of the operator from signal to the estimation error is conservative. Instead, we can use the a priori knowledge of positiveness of the signal by considering the same problem with l_∞^+ to l_∞ induced norm.

Based on this development, we consider the model matching problem to show that time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems. Also, synthesizing an LTI controller to optimize the l_∞^+ to l_∞ induced norm reduces to linear programming. We further generalize the results to the case of mixed input signals when there are inputs both in l_∞^+ and l_∞. As an example, we consider the aforementioned filtering problem and solve it when the signal is positive and bounded and there also exists noise which is only bounded but not necessarily positive.

In the second part of the paper, we address the cases where the positivity constraints are imposed on the systems. From the input–output perspective, an externally positive system is one whose output is in the positive l_∞ cone when the input is in this cone, starting from zero initial condition. As we point out, if such a constraint is imposed on the closed loop map, finding an optimal controller is a linear programming problem and hence tractable (Elia & Dahleh, 1998). Also, if the model matching problem for LTI systems is considered, time varying linear or nonlinear compensation cannot outperform LTI even if external positivity is enforced. Furthermore, if internal positivity is sought, we show that a dynamic controller offers no advantage over a static one as far as l_1, l_∞, or $\mathcal{H}_\infty$ performance is concerned. Therefore, the abovementioned results can be readily used to obtain an optimal (static) state feedback controller or output feedback for special cases. We note that, designing an optimal output feedback controller (which is static) is a harder problem and in general leads to a bilinear program. In certain cases, however, when the measurement matrix satisfies certain conditions, such problem is shown to reduce to a linear program.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote the sets of positive integers, non-negative integers, real numbers, positive real numbers, n-dimensional real vectors, and $n \times m$ dimensional real matrices, respectively. For any $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, its l_1 and l_∞ norms are defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$. For any $M = [m_{ij}] \in \mathbb{R}^{n\times m}$, $\|M\|_1 = \max_i \sum_{j=1}^m |m_{ij}|$, $\|M\|_\infty = \max_j \sum_{i=1}^n |m_{ij}|$, and its null space is denoted by $Null(M)$. For a full-row rank matrix $M \in \mathbb{R}^{n\times m}$, with $m \ge n$, let $N(M) \in \mathbb{R}^{m\times(m-n)}$ be a matrix whose columns span the null space of M. Also, associated with M, we define two matrices $M^+ = [m_{ij}^+] \in \mathbb{R}^{n\times m}$ and $M^- = [m_{ij}^-] \in \mathbb{R}^{n\times m}$ as

$$m_{ij}^+ = 0 \vee m_{ij}, \qquad m_{ij}^- = 0 \vee -m_{ij},$$

where $\vee$ stands for the max operator. That is, for two real numbers a and b, $a \vee b := \max\{a, b\}$. We refer to M^+ and M^- as the positive decomposition of M and it can be easily verified that $M = M^+ - M^-$. Given a sequence $y = \{y(k)\}_{k=1}^\infty$ where $y(k) \in \mathbb{R}^n$, for $k \in \mathbb{Z}_+$, one can define its positive decomposition into two non-negative sequences y^+ and y^- in an analogous way. Furthermore, its l_1 and l_∞ norms are given by $\|y\|_1 = \sum_{i=1}^\infty \|y(k)\|_1$ and $\|y\|_\infty = \sup_{k\in\mathbb{N}} \|y(k)\|_\infty$, whenever they are finite. The space of such sequences whose l_1 or l_∞ norm is finite is denoted by l_1^n and l_∞^n, respectively.

Note that l_∞^n is the space of bounded sequences. In this paper, we are also interested in a certain subset of l_∞^n which is denoted by l_∞^{n+}. This set is characterized as

$$l_\infty^{n+} = \left\{\{y(k)\}_{k=1}^\infty \in l_\infty^n : y_i(k) \ge 0, k \in \mathbb{Z}_+, i = 1, \dots, n\right\},$$

where $y_i(k)$ is the ith entry of vector $y(k) \in \mathbb{R}^n$. In other words, l_∞^{n+} is the set of bounded non-negative sequences. By $\mathcal{B}\left(l_\infty^{n+}, \varepsilon\right)$ $(\mathcal{B}(l_\infty^n, \varepsilon))$, for $\varepsilon > 0$, we mean the ball of radius ε in l_∞^{n+} (l_∞^n).

Let $\mathcal{L}_{TV}^{n\times m}$ be the space of all linear, causal, and bounded operators, $T : l_\infty^m \to l_\infty^n$. That is, for any $x, y \in l_\infty^m$, $T(x+y) = Tx + Ty$, $P_k T P_k u = T P_k u$, for $\forall k \in \mathbb{Z}_+$, and

$$\|T\| := \sup_{u\neq 0} \frac{\|Tu\|_\infty}{\|u\|_\infty} < +\infty, \tag{1}$$

where P_k is the truncation operator defined by

$$P_k x = (x_0, x_1, \dots, x_{k-1}, 0, 0, \dots).$$

Also, denote by $\mathcal{L}_{TI}^{n\times m}$ the subspace of all $T \in \mathcal{L}_{TV}^{n\times m}$ such that $\Lambda T = T\Lambda$, where Λ is the delay operator

$$\Lambda x = \Lambda(x_0, x_1, \dots) = (0, x_0, x_1, \dots), \quad \text{for } \forall x \in l_\infty^m.$$

It is well-known that any $T \in \mathcal{L}_{TV}^{n\times m}$ can be represented by a lower triangular infinite dimensional matrix

$$T = [T(i,j)]_{i\ge j} = \begin{bmatrix} T(0,0) & 0 & 0 & \cdots \\ T(1,0) & T(1,1) & 0 & \cdots \\ T(2,0) & T(2,1) & T(2,2) & \\ \vdots & & & \ddots \end{bmatrix}, \tag{2}$$

where $T(i,j) \in \mathbb{R}^{n\times m}$ for all $i, j \in \mathbb{Z}_+$, $i \ge j$. Moreover, (1) defines a norm on $\mathcal{L}_{TV}^{n\times m}$ and

$$\|T\| = \sup_{i\in\mathbb{Z}_+} \left\| \begin{bmatrix} T(i,0) & T(i,1) & \cdots & T(i,i) \end{bmatrix} \right\|_1. \tag{3}$$

Also, one can think of the positive decomposition of T into $T^+ = [T^+(i,j)]_{i\ge j} \in \mathcal{L}_{TV}^{n\times m}$ and $T^- = [T^-(i,j)]_{i\ge j} \in \mathcal{L}_{TV}^{n\times m}$.

In Shamma and Dahleh (1991), the authors introduced the normed space $\mathcal{L}_0^{m\times n}$ whose elements, $G \in \mathcal{L}_0^{m\times n}$, can be represented by upper triangular infinite dimensional matrices

$$G = \begin{bmatrix} G(0,0) & G(0,1) & G(0,2) & \cdots \\ 0 & G(1,1) & G(1,2) & \cdots \\ 0 & 0 & G(2,2) & \\ \vdots & & & \ddots \end{bmatrix},$$

where $G(i,j) \in \mathbb{R}^{m\times n}$ for all $i, j \in \mathbb{Z}_+$ and $j \ge i$. Moreover, $\mathcal{L}_0^{m\times n}$ is equipped with a norm, $\|.\|_{\mathcal{L}_0}$,

$$\|G\|_{\mathcal{L}_0} = \sum_i \|\mathcal{C}[G]_i\|_\infty,$$