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Brief paper

Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks ☆

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ABSTRACT

In-depth understanding of the spectral properties of grounded Laplacian matrices is critical for the analysis of convergence speeds of dynamical processes over complex networks, such as opinion dynamics in social networks with stubborn agents. We focus on grounded Laplacian matrices for directed graphs and show that their eigenvalues with the smallest real part must be real. Lower and upper bounds for such eigenvalues are provided utilizing tools from nonnegative matrix theory. For those eigenvectors corresponding to such eigenvalues, we discuss two cases when we can identify the vertex that corresponds to the smallest eigenvector component. We then discuss an application in leader–follower social networks where the grounded Laplacian matrices arise naturally. With the knowledge of the vertex corresponding to the smallest eigenvector component for the smallest eigenvalue, we prove that by removing or weakening specific *directed* couplings pointing to the vertex having the smallest eigenvector component, all the states of the other vertices converge faster to that of the leading vertex. This result is in sharp contrast to the well-known fact that when the vertices are connected together through *undirected* links, removing or weakening links does not accelerate and in general decelerates the converging process.
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1. Introduction

The spectral properties of certain matrices of a given network topology graph reveal ample information on the structures of the corresponding network. The study on those spectral properties plays an important role in the analysis of the convergence and convergence speed of the dynamical process evolving on such networks. In the study of multi-agent networks (Cao, Morse, & Anderson, 2008; Jadbabaie, Lin, & Morse, 2003; Ni & Cheng, 2010; Ren & Beard, 2005; Scardovi & Sepulchre, 2009; Xia & Cao, 2011, 2014), researchers have been especially interested in the process of aligning followers with the leaders when some agents are taking the role of leaders that guide the followers to reach consensus (Cao et al., 2008; Jadbabaie et al., 2003; Ni & Cheng, 2010; Scardovi & Sepulchre, 2009); similarly, in the study of social networks (Acemoglu, Como, Fagnani, & Ozdaglar, 2013; Blondel, Hendrickx, & Tsitsiklis, 2009; Ghaderi & Srikant, 2012; Xia, Cao, & Johansson, 2016; Yildiz, Acemoglu, Ozdaglar, Saberi, & Scaglione, 2011), people have also studied the process of opinion forming in the presence of stubborn agents that keep their opinions unchanged over time (Acemoglu et al., 2013; Ghaderi & Srikant, 2012; Yildiz et al., 2011). In such cases, the grounded Laplacian matrices (Bollobas, 1998; Miekkala, 1993) obtained by removing the rows and columns corresponding to the leaders or stubborn agents in the Laplacian matrices become critical in determining the convergence and the convergence rate of the system. The spectral properties of grounded Laplacian matrices are especially useful for the stability analysis of multi-agent formations (Barooah & Hespanha, 2006).

For undirected graphs, the spectral properties of grounded Laplacian matrices have been investigated, where upper and lower bounds have been established for their smallest eigenvalues; in particular, a special class of graphs, i.e., random graphs, have been discussed (Pirani & Sundaram, 2014, 2016). In the study of synchronization of complex networks, great efforts have been

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devoted to identifying which vertices in a network should be controlled and what kinds of controllers should be designed to achieve synchronization and to optimize the convergence speed (Shi, Sou, Sandberg, & Johansson, 2014; Yu, Chen, & Lü, 2009).

Although the study on the spectral properties of Laplacian matrices and grounded Laplacian matrices for undirected graphs is fruitful, the counterpart for directed graphs is limited (Agaev & Chebotarev, 2005; Hao & Barooah, 2011) and some of the established results for undirected graphs do not carry over to the directed case. In this paper, we study the spectral properties of the grounded Laplacian matrices for directed graphs and look into their applications. Since the graphs are directed, the results, such as Rayleigh quotient inequality and the interlacing theorem for deriving some bounds for symmetric Laplacian matrices of undirected graphs in Pirani and Sundaram (2014, 2016), do not apply. We resort to nonnegative matrix theory and show that the eigenvalue with the smallest real part of the directed Laplacian matrix is real and the bounds established in Pirani and Sundaram (2014) still hold for this eigenvalue. The properties of the eigenvector corresponding to this eigenvalue of the directed Laplacian matrix are also discussed. In addition, two specific cases are identified when one can tell which vertex corresponds to the smallest eigenvector component.

We then discuss an application to leader–follower networks in multi-agent systems. With the knowledge of the vertex whose eigenvector component for the smallest eigenvalue is the smallest, we study the problem of accelerating the process of reaching consensus in a network with leaders. We propose a new strategy based on weakening the weights of or removing some specific edges. Although in *undirected* multi-agent networks, stronger or more links between followers often accelerate convergence (Xiao & Boyd, 2004), in *directed* networks, the convergence speed changes in more complicated fashions (Cao et al., 2008; Cao, Olshevsky, & Xia, 2014). We claim that if we cut or weaken the links that point from the other followers to that follower corresponding to the smallest eigenvector component, the convergence process of *all* the followers may get accelerated.

The rest of the paper is organized as follows. In Section 2, we introduce grounded Laplacian matrices and give some preliminaries on nonnegative matrices. In Section 3, we establish the bounds for the eigenvalue with the smallest real part of the grounded Laplacian matrix and discuss the properties of its corresponding eigenvector. Section 4 identifies two cases when we can tell which vertex corresponds to the smallest eigenvector component. Section 5 discusses the applications of grounded Laplacian matrices in leader–follower networks.

2. Grounded Laplacian matrices for directed networks

Consider a directed network consisting of $N > 1$ vertices whose topology is described by a directed, positively weighted graph $\mathbb{G}$. Let $A = (a_{ij})_{N\times N}$ be the adjacency matrix for $\mathbb{G}$, and then a_{ij}, $1 \le i, j \le N$, is nonzero if and only if there is a directed edge from vertex j to i in $\mathbb{G}$ in which case a_{ij} is exactly the positive weight of the edge (j, i). Let $d_i = \sum_{j=1, j\neq i}^{N} a_{ij}$ be the *in-degree* of each vertex i and associate $\mathbb{G}$ with the diagonal *degree matrix* $D = \text{diag}\{d_1, d_2, \ldots, d_N\}$. Then the *Laplacian matrix* for the positively weighted, direct graph $\mathbb{G}$ is defined by $L = D - A$. It is well known that the spectral properties of the Laplacian matrix L can be conveniently studied when taking the network to be an N-vertex electrical network where each a_{ij} corresponds to the resistance from vertex j to i and some vertices are taken to be the source and some others the sink of the electrical current flowing in the network (Bollobas, 1998, Chap 2). In this context, it is of particular interest to study the case when some vertices are grounded. Let $\mathcal{V} = \{1, \ldots, N\}$ denote the set of indices of all the vertices and $\mathcal{S} = \{n+1, \ldots, N\}$ for some $1 < n < N$ be the set of indices of all the grounded vertices. Then the Laplacian matrix can be partitioned into

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_g & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \tag{1}$$

where the rows and columns of L_{22} correspond to the vertices in $\mathcal{S}$ and the $n \times n$ submatrix L_{11} is called the *grounded Laplacian matrix* (Miekkala, 1993) and we denote it in the rest of the paper by L_g.

The grounded Laplacian matrices have some special properties and it is the main goal of this paper to study their spectral properties. But before doing that, we first summarize and prove some useful general results for matrix analysis.

Let $M = (m_{ij})_{N\times N}$ be a real matrix. We write $M \ge 0$ if $m_{ij} \ge 0$, $i, j = 1, \ldots, N$, and such a matrix M is called a *nonnegative* matrix. It is straightforward to check that the grounded Laplacian matrices are *not* nonnegative, but later we will show how to transform a grounded Laplacian matrix into a nonnegative matrix. We denote the spectral radius of M by $\rho(M)$. It follows from the Perron–Frobenius theorem (Horn & Johnson, 1985) that for a nonnegative matrix M, $\rho(M)$ is an eigenvalue of M and there is a nonnegative vector $x \ge 0$, $x \ne 0$, such that $Mx = \rho(M)x$. In addition, if M is irreducible, then $\rho(M)$ is a simple eigenvalue of M and there is a positive vector $x > 0$ such that $Mx = \rho(M)x$.

Lemma 1. *Suppose that $M \in \mathbb{R}^{N\times N}$ is an irreducible nonnegative matrix and $\min_{1\le i\le N} \sum_{j=1}^{N} m_{ij} < \max_{1\le i\le N} \sum_{j=1}^{N} m_{ij}$. Then*

$$\min_{1\le i\le N} \sum_{j=1}^{N} m_{ij} < \rho(M) < \max_{1\le i\le N} \sum_{j=1}^{N} m_{ij}. \tag{2}$$

Proof. Let $\alpha = \max_{1\le i\le N} \sum_{j=1}^{N} m_{ij}$ and construct a new matrix B with $b_{ij} = \alpha \frac{m_{ij}}{\sum_{j=1}^{N} m_{ij}}$. Then $B \ge M$, and $\sum_{j=1}^{N} b_{ij} = \alpha$ for all $i = 1, \ldots, N$, implying $\rho(B) = \alpha$. Since $B - M \ge 0$, $B - M \ne 0$, and M is irreducible, from Problem 15 in pp. 515 in Horn and Johnson (1985), one knows $\rho(M) < \rho(B) = \alpha$. The lower bound can be established in a similar manner. □

Lemma 2. *Let $M \in \mathbb{R}^{N\times N}$ be an irreducible nonnegative matrix. Then for any positive vector x we have*

$$\min_{1\le i\le N} \frac{(Mx)_i}{x_i} \le \rho(M) \le \max_{1\le i\le N} \frac{(Mx)_i}{x_i}, \tag{3}$$

where $(Mx)_i$ is the ith element of the vector Mx. There is a unique vector $x^* \in \{x | x > 0,\ x^T x = 1\}$ such that $\rho(M) = \frac{(Mx^*)_i}{x_i^*}$, $i = 1, \ldots, N$, and for any $y \in \{x | x > 0,\ x^T x = 1\}$, $y \ne x^*$,

$$\min_{1\le i\le N} \frac{(My)_i}{y_i} < \rho(M) < \max_{1\le i\le N} \frac{(My)_i}{y_i}. \tag{4}$$

Proof. Inequality (3) is Theorem 8.1.26 in Horn and Johnson (1985). Since M is nonnegative and irreducible, there is a unique vector $x^* \in \{x | x > 0,\ x^T x = 1\}$ such that $Mx^* = \rho(M)x^*$, which implies $\rho(M) = \frac{(Mx^*)_i}{x_i^*}$, $i = 1, \ldots, N$.

Since M^T is nonnegative and irreducible, there is a positive vector $z > 0$ such that $M^T z = \rho(M)z$. Now we prove (4) by contradiction. Suppose there is another vector $y \in \{x | x > 0,\ x^T x = 1\}$, $y \ne x^*$, such that $\rho(M) = \min_{1\le i\le N} \frac{(My)_i}{y_i}$. Thus $\rho(M)y_i \le (My)_i$ for all $i = 1, \ldots, N$, namely $My - \rho(M)y \ge 0$. Then

$$z^T(My - \rho(M)y) = \rho(M)z^T y - \rho(M)z^T y = 0.$$

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