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Brief paper

Almost equitable partitions and new necessary conditions for network controllability[☆]

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ABSTRACT

In this paper, we consider the controllability problem for multi-agent networked control systems. The main results of the paper are new graph-theoretic necessary conditions for controllability involving almost equitable graph vertex partitions. We generalize the known results on the role of graph symmetries and uncontrollability to weighted digraphs with multiple-leaders and we also consider the broadcasted control scenario. Our results show that the internal structure of communities in a graph can induce obstructions to controllability that cannot be characterized by symmetry arguments alone and that in some cases depend on the number-theoretic properties of the communities. We show via examples that our results can be used to account for a large portion of uncontrollable inducing leader-selections that could not have otherwise been accounted for using symmetry results.

1. Introduction

The *controllability* property of a controlled dynamical system is one of the central notions in control systems theory. Roughly speaking, the controllability problem is concerned with whether it is possible to transfer the state of a controlled dynamic system from some given initial condition to any final desired state. For linear control systems, there are several equivalent characterizations of controllability and it is well-known that the property is generic in the sense that the set of systems that are controllable form an open and dense subset of the space of all system parameters. Despite the generic nature of controllability for linear systems, there has been recent interest in the control community to understand the controllability property for linear multi-agent networked control systems, see for instance Aguilar and Gharesifard (2015a), Chapman, Nabi-Abdolyousefi, and Mesbahi (2014), Martini, Egerstedt, and Bicchi (2010), Monshizadeh, Zhang, and Camlibel (2014), Notarstefano and Parlangeli (2013), Parlangeli and Notarstefano (2012), Rahmani, Ji, Mesbahi, and Egerstedt (2009), Tanner (2004) and Zhang, Cao, and Camlibel (2014) and references therein. The goal of this recent research effort is to understand how graph-theoretic properties of the underlying network relate to the controllability property of the linear control system whose system matrix encodes the adjacency relationships in the network (e.g., adjacency matrix, Laplacian matrix, etc.).

It is natural to ask why such an effort has been devoted to characterize a generic property such as controllability for a linear control system. To answer this question, we first note that the generic nature of controllability when system parameters are allowed to vary *continuously* does not in any trivial way imply that controllability is also generic for *discrete* combinatorial objects such as graphs. The situation is similar, for instance, when considering the question of the likelihood that a matrix will have simple eigenvalues. If the matrix entries are allowed to vary continuously then the set of real matrices with simple eigenvalues forms a dense and open subset of the associated Euclidean space. In contrast, up until very recently and using sophisticated machinery, it was shown that the adjacency matrix of a graph will almost surely have simple eigenvalues as the size of the graph increases (Tao & Vu, 2014), and thus settling a conjecture posed by L. Babai in the 1980s. Similarly, it was also shown very recently that the controllability property for the adjacency matrix of a graph, and in the special case where all nodes are controlled, is generic, again in the sense that the proportion of controllable systems tends to one as the size of the graph increases (thus settling a conjecture posed by Godsil, 2012). Second of all, it has been documented in

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MacArthur, Sánchez-García, and Anderson (2008) that many real-world biological, technological, and social networks possess a high level of symmetry and, as it was shown in Rahmani et al. (2009), symmetries are obstructions to controllability for networked multi-agent control systems. As pointed out in MacArthur et al. (2008), many real-world networks contain "tree-like" symmetries due to the fact that these networks are grown from existing vertices and this growth process introduces branch-like structures. Since almost all tree networks have symmetries (Erdös & Réyni, 1963), it is not too surprising then that such symmetries would be present in these networks. Hence, although symmetry-like structures are mathematically rare, they seem to be ubiquitous in many real-world networks and therefore it is important to uncover what other symmetry-like structures induce uncontrollability.

The purpose of this paper is to identify a symmetry-like structure, specifically a class of graph vertex partitions, that when present in a network can obstruct controllability in a multi-agent control system even when the control nodes have been selected to "break" all symmetries. It is well-known that symmetries are not necessary for uncontrollability (Rahmani et al., 2009) but aside from special classes of graphs such as trees, grid graphs, threshold graphs, and circulant graphs (Aguilar & Gharesifard, 2015b; Ji, Lin, & Yu, 2012; Nabi-Abdolyousefi & Mesbahi, 2013; Notarstefano & Parlangeli, 2013), and a linear–algebraic characterization of controllability for multi-agent systems (Chapman & Mesbahi, 2014), little is known about what general *intrinsic graph-structures* induce uncontrollability. A purpose of this paper is to narrow this gap.

As mentioned, we identify a class of graph vertex partitions, and in particular a class of *almost equitable partitions* (Cardoso, Delorme, & Rama, 2007), that induce uncontrollability in a non-trivial way. Moreover, these partitions can account for a significant portion of the uncontrollable leader selections that could not have been detected by symmetry arguments alone. Using graph vertex partitions to study control-theoretic properties in multi-agent systems is not new and in fact is becoming an increasingly useful tool in the analysis and design of multi-agent control systems, see for instance Martini et al. (2010), Monshizadeh, Trentelman, and Camlibel (2014), Monshizadeh, Zhang, and Camlibel (2015), Rahmani et al. (2009), Zhang et al. (2014) and references therein. Graph vertex partitions also take an important role in the study of synchrony and pattern formation in coupled cell networks (Golubitsky, Stewart, & Török, 2005; Stewart, Golubitsky, & Pivato, 2003).

Statement of contributions

The main results of this paper are new general graph-theoretic necessary conditions for controllability of leader–follower Laplacian dynamics using almost equitable graph partitions (Theorems 3, 4, 5). Our first main result (Theorem 3) generalizes the relationship between graph symmetries and uncontrollability in the multi-input case (Rahmani et al., 2009) to also include the scenario of multi-input broadcast control. Roughly speaking, this result shows that if the nodes are selected to respect the structure of a non-trivial almost equitable partition then the resulting control system is uncontrollable. Our next main result (Theorem 4) identifies a new obstruction to controllability that we call *reducible equitable partitions*. These partitions are, to the best knowledge of the authors, the first general and intrinsic graph-theoretic structure that induce uncontrollability unrelated to graph symmetries. In other words, when these partitions are present, one might have chosen the leaders to break all "symmetries" induced by all non-trivial almost equitable partitions but yet the induced dynamics are uncontrollable. The result demonstrates that the internal structure of the cells of a partition can also introduce undesirable controllability properties. Our last main result (Theorem 5) uses the quotient graph induced by an almost equitable partition to identify a new obstruction to controllability that again cannot be captured by symmetry or using reducible equitable partitions. This result also demonstrates that the internal structure of the cells of a partition must be taken into account in controllability analysis. Overall, our results provide explicit graph-theoretic structures that when present in a network can induce uncontrollable dynamics.

Although we consider Laplacian dynamics, our results are applicable to other graph matrices that encode the adjacency structure of a graph. Also, we focus on the broadcasted control problem since it allows us to broaden the view of how symmetries, and more generally equitable partitions, induce uncontrollability and allows us to identify new obstructions to controllability that could not have been captured otherwise. We note that the broadcasted multi-agent control scenario is also considered in Godsil (2012) and Yoon, Rowlinson, Cvetković, and Stanić (2014).

2. Preliminaries and problem statement

Throughout this paper, the standard basis vectors in $\mathbb{R}^n$ are denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The orthogonal complement of a set $S \subset \mathbb{R}^n$ under the standard inner product on $\mathbb{R}^n$ will be denoted by $S^\perp$. The transpose of a matrix $\mathbf{M}$ is denoted by $\mathbf{M}^T$. The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$. The conjugate transpose of $\mathbf{w} \in \mathbb{C}^n$ is denoted by $\mathbf{w}^*$. The column space and null space of a matrix $\mathbf{M}$ will be denoted by $\mathrm{img}(\mathbf{M})$ and $\ker(\mathbf{M})$, respectively. The cardinality of a finite set C will be denoted by $|C|$. The greatest common divisor of a set of integers $k_1, k_2, \ldots, k_r$ is denoted by $\gcd(k_1, k_2, \ldots, k_r)$.

2.1. Permutations

We denote the symmetric group on $V = \{1, 2, \ldots, n\}$ by S_n, i.e., S_n is the group of permutations $\sigma : V \to V$. It is well-known (Dummit & Foote, 1991) that each $\sigma \in S_n$ has a unique (up to ordering) cycle decomposition of the form

$$\sigma = \underbrace{(i_1\ i_2 \cdots i_{m_1})}_{\rho_1} \underbrace{(i_{m_1+1}\ i_{m_1+2} \cdots i_{m_2})}_{\rho_2} \cdots \underbrace{(i_{m_{r-1}+1}\ i_{m_{r-1}+2} \cdots i_{m_r})}_{\rho_r},$$

where ρ_j is the permutation that cyclically permutes the integers $i_{m_{j-1}+1}, i_{m_{j-1}+2}, \ldots, i_{m_j}$ and fixes all other integers. The set of integers $C_j = \{i_{m_{j-1}+1}, i_{m_{j-1}+2}, \ldots, i_{m_j}\}$ is called a *cell*.

The set of binary vectors of length n will be denoted by $\{0, 1\}^n$. The *characteristic indices* of $\mathbf{b} \in \{0, 1\}^n$, denoted by $\chi(\mathbf{b})$, is the subset of indices where $\mathbf{b}$ is non-zero. For instance, if $\mathbf{b} = [0\ 1\ 1\ 0\ 0\ 1]^T \in \{0, 1\}^6$ then $\chi(\mathbf{b}) = \{2, 3, 6\}$. Conversely, given any set $C \subseteq \{1, 2, \ldots, n\}$ of indices, the *characteristic vector* of C is the unique vector $\mathbf{c} \in \{0, 1\}^n$ such that $\chi(\mathbf{c}) = C$. The all ones vector and the zero vector will be denoted by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively.

2.2. Graphs

Our notation from graph theory is standard and any basic notion not defined here can be found in Godsil and Royle (2001). In this paper, we consider graphs that may be weighted and/or directed, but not containing loops. For a (directed) graph G, or *digraph*, we denote by $V(G)$ its vertex set and by $E(G) \subseteq V(G) \times V(G)$ its edge set. We consider only finite graphs and so we assume throughout that $V(G) = \{1, 2, \ldots, n\}$. The weight of the edge $(i, j) \in E(G)$ will be denoted by $a_{ij} \in \mathbb{R}$, and if $(i, j) \notin E(G)$ we set $a_{ij} = 0$. When not explicitly stated, we assume that G is *strongly connected*.