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Brief paper

# Design of Continuous Twisting Algorithm ☆

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ABSTRACT

For the double integrator with matched Lipschitz disturbances we propose a continuous homogeneous controller providing finite-time stability of the origin. The disturbance is compensated exactly in finite time using a discontinuous function through an integral action. Since the controller is dynamic, the closed loop is a third order system that achieves a third order sliding mode in the steady state. The stability and robustness properties of the controller are proven using a smooth and homogeneous strict Lyapunov function (LF). In a first stage, the gains of the controller and the LF are designed using a method based on Pólya's Theorem. In a second stage the controller's gains are adjusted through a sum of squares representation of the LF.



## 1. Introduction

Robustness is one of the main issues in control systems theory. Hence, several approaches to deal with system's disturbances have been developed. Sliding Mode Control has become one of the most efficient techniques to control uncertain plants under non-vanishing disturbances (Edwards & Spurgeon, 1998; Utkin, Guldner, & Shi, 2009). Such controllers are able to compensate, theoretically exactly, matched disturbances by confining the system's trajectories in a properly chosen sliding surface. In general, this is achieved using discontinuous controllers with theoretical infinite switching frequency (Ding, Levant, & Li, 2016; Levant, 2003; Utkin et al., 2009). In this paper we consider the disturbed double integrator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u + \Delta(t), \tag{1}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the control input and $\Delta(t) \in \mathbb{R}$ is a disturbance. We assume that $\Delta$ is a Lipschitz function, thus, its derivative exists for almost all $t \geq 0$ and is uniformly bounded, i.e., $|\dot{\Delta}(t)| \leq \mu$ for a known $\mu \in \mathbb{R}$. The problem is to drive the state $x$ to the origin in finite-time by using a continuous control signal despite the disturbance $\Delta$. Below we list the reasons whereby most of the known Sliding Mode controllers (Conventional or Higher-Order) are unable to solve the stated problem.

- Conventional Sliding Mode controller (Utkin et al., 2009). This controller is discontinuous and a sliding variable with relative degree one must be designed. The convergence of the state to the sliding surface is in finite time but exponential to the origin. This controller can reject only bounded disturbances.
- Second Order Sliding Mode (SOSM) controller, Super-Twisting Algorithm (STA) (Levant, 1993). To apply the STA to (1), it is necessary to design a sliding variable with relative degree one. STA has two advantages, the control signal is continuous, and the Lipschitz disturbances can be rejected. However the convergence rate of the state to the origin is exponential.
- SOSM controller, Twisting Algorithm (Levant, 1993). This controller does not require sliding variable design, and provides finite time convergence of the state to the origin. However, the control signal is discontinuous, and the disturbances must be bounded. It is worth to recall that this homogeneous controller is capable to ensure convergence of $x_1$ and $x_2$ to zero in finite time despite of a bounded perturbation. Moreover, under discretization, such controller guarantees quadratic precision of $x_1$ with respect to the sampling period (Levant, 1993).
- Third Order Sliding Mode controllers by means of the introduction of a virtual state (Bartolini, Ferrara, & Usai, 1998; Levant,

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1993). The controllers of this kind provide a continuous control signal, and can avoid the designing of a sliding variable. Although they are able to achieve a second order sliding mode, they require, as information, $x_1, x_2$ and $\dot{x}_2$. This is also the case of the smoothed Integral High-Order Sliding Modes (Levant & Alelishvili, 2007).

Recently, in Edwards and Shtessel (2016), a continuous Sliding Mode controller was proposed. However, since such algorithm is not homogeneous, the issue of precision requires more investigation. A new class of homogeneous continuous Sliding Mode controllers (based on a generalization of the STA) for systems with relative degree two was announced (Chitour, Harmouche, & Laghrouche, 2016; Fridman, Moreno, Bandyopadhyay, Kamal, & Chalanga, 2015; Kamal, Chalanga, Moreno, Fridman, & Bandyopadhyay, 2014; Kamal, Moreno, Chalanga, Bandyopadhyay, & Fridman, 2016; Moreno, 2016; Zamora, Moreno, & Kamal, 2013). These new controllers are homogeneous of negative degree with weights $r_1 = 3$ for $x_1$ and $r_2 = 2$ for $x_2$. This property ensures cubic and quadratic precision, respectively, with respect to the sampling period (Levant, 1993).

In this paper we propose a controller (announced in Torres-Gonzalez, Fridman, & Moreno, 2015) to solve the problem stated above. Since the part of the algorithm ensuring disturbance compensation has a structure as the Twisting algorithm, it is called Continuous Twisting Algorithm (CTA). Such controller has the following advantages:

- The generated control signal is continuous;
- Lipschitz disturbances can be compensated;
- The states $x_1$ and $x_2$ converge to zero in finite time;
- The closed loop achieves a third-order sliding mode with respect to the system's states and the state of the controller;
- Under discretization, the controller ensures cubic and quadratic precision with respect to the sampling period for $x_1$ and $x_2$, respectively.

For the controller of the present paper, we provide a LF designed with the methodology given in Sanchez and Moreno (2014), that consists in proposing a *generalized form* (GF) as a candidate LF. The positive definiteness of the candidate LF (and the negative definiteness of its derivative along the system's trajectories) is verified by using the idea in the Pólya's theorem (Hardy, Littlewood, & Pólya, 1988; Pólya, 1928). In this paper we perform a second step to adjust the controller's gains by using a sum of squares (SOS) representation of the LF. An important reason to use the SOS procedure is that it reduces the problem of designing the controller's gains to an LMI problem. Moreover, the procedure allows to include optimization criteria, in this case, we have maximized the bound of the admitted disturbances. In this paper we use the software SOSTOOLS (Prajna, Papachristodoulou, & Parrilo, 2002–2005) to solve such LMI/optimization problem.

Although there exist some other procedures to design LFs for High Order Sliding Modes (Polyakov & Poznyak, 2012; Sanchez & Moreno, 2012), they are very difficult to apply in the present case and generally provide non-smooth LFs. Moreover, they are useful only for SOSM (Polyakov & Poznyak, 2012) or require the solutions of the system (Sanchez & Moreno, 2012). Unlike those methods, GFs approach is quite more general in the sense that for any homogeneous system (of any order) described by GFs the method provides a systematic procedure to search for a differentiable LF that is also a GF. Thus, since our controller is described by GFs, this method is very suitable.

It is important to mention that the controllers provided in Kamal et al. (2016) and Moreno (2016) have similar properties as those of the CTA. They also possess differentiable LFs, however, the designing of their gains consist in solving nonlinear inequalities in the parameters of the LFs and the controller's gains. Unfortunately such controllers are not described by GFs, so that, they cannot be analysed and designed directly following the procedure used in the present work.

This paper is organized as follows. In Section 2 we present the CTA controller and a smooth LF. We also discuss some features of the controller and a procedure to design its gains. Section 3 shows the procedure to design the LF and the computation of some controller's gains. In Section 4 some numerical simulations are presented. Finally in Section 5 some concluding comments about the CTA are given.

## 2. Main results

### 2.1. The controller

To solve the stated problem in the previous section we propose the following dynamic controller

$$\begin{aligned} u(x) &= -k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_2\rfloor^{\frac{1}{2}} + \eta \\ \dot{\eta} &= -k_3\lceil x_1\rfloor^{0} - k_4\lceil x_2\rfloor^{0}, \end{aligned} \tag{2}$$

where the notation $\lceil\cdot\rfloor^{\gamma} = |\cdot|^{\gamma}\operatorname{sign}(\cdot)$ was used. The reals $k_i > 0$, $i = 1, \ldots, 4$, are parameters to be designed. For simplicity we define the vector of parameters $k = [k_1, \ldots, k_4]^{\top}$. Notice that the right hand side on the second equation in (2) has the structure of the Twisting controller (Levant, 1993), this is integrated through $\eta$ generating a continuous signal that allows the controller to reject a perturbation with bounded derivative. The following theorem constitutes one of the main results of this paper.

**Theorem 1.** *For any positive real $\mu < \infty$, $x = 0$ is a finite time stable equilibrium point of (1) with the controller (2) for properly designed gains $k_i$, $i = 1, \ldots, 4$.*

The proof of this theorem is Lyapunov based and it is given in Section 4. The methodology for the adjustment of the controller's gains is presented in Section 4. However in Section 2.4 we provide some examples to design the parameters of the controller.

### 2.2. Lyapunov function

The closed loop (1), (2) exhibits useful homogeneity properties that we exploit in this paper. In Appendix A we recall some definitions about homogeneity. Define the virtual state $x_3 \triangleq \eta + \Delta(t)$, by combining (2) and (1) the closed loop system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_2\rfloor^{\frac{1}{2}} + x_3 \\ \dot{x}_3 &= -k_3\lceil x_1\rfloor^{0} - k_4\lceil x_2\rfloor^{0} + \dot{\Delta}(t). \end{aligned} \tag{3}$$

The third equation of (3) is discontinuous and uncertain due to the sign functions and the disturbance term. Thus, it can be associated with the differential inclusion (DI) $\dot{x}_3 \in -k_3\lceil x_3\rfloor^0 - k_4\lceil x_4\rfloor^0 + [-\mu, \mu]$, (sign function is defined in Appendix A). Therefore, (3) is associated with the DI $\dot{x} \in F(x)$ where the set valued map $F$ is given by $F(x) = \{y \in \mathbb{R}^n : y = [x_2, x_3, \rho]^{\top}\}$, for all $\rho \in \{-k_3\lceil x_3\rfloor^0 - k_4\lceil x_4\rfloor^0 + [-\mu, \mu]\} \subset \mathbb{R}$. This DI is homogeneous of degree $q = -1$ with weights $\mathbf{r} = [3, 2, 1]^{\top}$. Hence, in this paper, the solutions of (3) are understood in the sense of Filippov (1988).

In nominal case, i.e., $\mu = 0$, the functions in the vector field of (3) are a special class of functions that have interesting properties. Classically, a homogeneous polynomial function is called *form*. In Sanchez and Moreno (2014) such set of functions was extended in order to include the kind of functions as those in the vector field of (3). Those functions are called GFs. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a GF of degree $m$ if: (a) it is a homogeneous function of degree $m$ for some vector of weights $\mathbf{r}$; (b) it consists of sums, products and sums of