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Brief paper

Necessary and sufficient stability conditions for linear systems with pointwise and distributed delays ☆

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ABSTRACT

A stability criterion for the exponential stability of systems with multiple pointwise and distributed delays is presented. Conditions in terms of the delay Lyapunov matrix are obtained by evaluating a Lyapunov–Krasovskii functional with prescribed derivative at a pertinent initial function that depends on the system fundamental matrix. The proof relies on properties connecting the delay Lyapunov matrix and the fundamental matrix, which are proven to be valid for both stable and unstable systems. The conditions are applied to the determination of the exact stability region for some examples.

1. Introduction

One of the best known results of control theory is without doubt the exponential stability criteria for linear time-invariant systems of ordinary differential equations, namely, the positive-definiteness of the Lyapunov matrix, solution of the Lyapunov equation, that defines the quadratic Lyapunov function. This paper is devoted to an extension of this result to the case of time-delay systems with multiple pointwise and distributed delays.

This is done in the Lyapunov–Krasovskii framework (Krasovskii, 1956), where the general results of the Lyapunov theory are extended to systems with delays by using, instead of functions, functionals that capture the whole state of the delay system.

This framework is widely used in the derivation of sufficient stability conditions obtained via the proposal of functionals of prescribed form, a topic that has been a fertile field of investigation in the past decades, due to the formulation of the conditions in terms of easy to test Linear Matrix Inequalities (see for example Fridman, 2014; Niculescu, 2001, and the references therein).

The converse approach, that consists of finding the form of the functional with prescribed derivative along the system trajectory was first addressed in Repin (1965). A detailed account of the results of the early contributors on the topic, Datko (1972), Huang (1989), Infante and Castelan (1978), and Louisell (2001), is given in Kharitonov (2013, p. 73).

In Kharitonov and Zhabko (2003), a class of complete type functionals that admit a quadratic lower bound if the corresponding system is exponentially stable was introduced. These functionals can be considered as a generalization of the quadratic Lyapunov functions, usually used for ordinary differential equations. They are defined by a matrix-valued function over the delay interval, which is the analogue of the Lyapunov matrix, and is the solution of a set of three properties that play the role of the Lyapunov equation. The reader is referred to Kharitonov (2013) for a comprehensive treatment of this topic, for different classes of delay systems; Some aspects of the computation of the Lyapunov matrices have been addressed in Huesca, Mondié, and Santos (2009), Jarlebring, Vanbiervliet, and Michiels (2011) and Kharitonov (2013).

Complete type functionals have been applied successfully to the robust stability analysis (Kharitonov & Zhabko, 2003), exponential estimates of solutions (Kharitonov & Hinrichsen, 2004), solution of the Bellman equation for time delay systems (Santos, Mondié, & Kharitonov, 2009), computation of the norm of the transfer matrix (Jarlebring et al., 2011), proof of the stability of predictor-based

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control scheme for state and input delay systems (Kharitonov, 2014).

It should be noted that these applications rely at some point on some stability assumption. It was only until recently that the direct application of complete type functionals to the stability analysis has received due attention. A criterion of the exponential stability of the scalar single delay equation has been given in Mondié (2012) and in Egorov and Mondié (2013). Some new sufficient stability conditions based on Lyapunov functionals of complete type were given in Medvedeva and Zhabko (2013). In the past few years, the complete type functionals have been used in order to derive families of necessary stability conditions for linear systems with multiple delays (Egorov & Mondié, 2014a,b), and with distributed delays (Cuvas & Mondié, 2015).

The visible efficacy of the above mentioned necessary exponential stability conditions in the accurate determination of the stability region of a variety of examples, added to the availability of a criterion for the scalar single delay case, points naturally towards a proof of sufficiency of these conditions. A preliminary result in this direction was presented in Egorov (2014).

The main contribution of the present paper is to prove necessary and sufficient conditions for the exponential stability of linear systems with multiple pointwise and distributed delays. The proof of this criterion relies on the presentation of fundamental stability/unstability results for the Lyapunov–Krasovskii functionals we introduce, the determination of the class of initial functions that reveal the role of the delay Lyapunov matrix, and on the approximation of any initial function by an element in this class.

The organization of the paper is as follows: Section 2 is devoted to some preliminaries on the delay systems under consideration. In Section 3, well known results on Lyapunov–Krasovskii functionals with prescribed derivative are recalled, and some new ones are introduced. The choice of initial functions leading to an expression of the functional in terms of the delay Lyapunov matrix, without assumption of stability of the system, is discussed in Section 5. The main result with its proof is given in Section 6. The paper ends with illustrative examples in Section 7, followed by concluding remarks.

Notation: The notations $Q > 0, Q \geqslant 0, Q \ngeqslant 0$, mean that the symmetric matrix Q is positive definite, positive semidefinite, and not positive semidefinite, respectively. The square block matrix with ith row and jth column element A_{ij} is written as $\{A_{ij}\}_{i,j=1}^{r}$. The symbol $*$ indicates transposed terms in symmetric block matrices.

2. Preliminaries

We consider linear systems of the form

$$\dot{x}(t) = \sum_{j=0}^{m} A_j x(t-h_j) + \int_{-H}^{0} G(\theta)x(t+\theta)d\theta, \tag{1}$$

where $t \geqslant 0$, $A_j, j = 0, \dots, m$, are real $n \times n$ matrices, delays are ordered as $0 = h_0 < h_1 < \cdots < h_m = H$, and $G(\theta)$, is a real piecewise continuous matrix function valued on $\theta \in [-H, 0]$. The initial function φ is assumed to be piecewise continuous, $\varphi \in \mathcal{H} = PC([-H, 0], \mathbf{R}^n)$, i.e., it has a finite number of discontinuity points of the first kind. The restriction of the solution $x(t, \varphi)$ of system (1) to the interval $[t-H, t]$ is denoted by

$$x_t(\varphi) : \theta \to x(t+\theta, \varphi), \quad \theta \in [-H, 0].$$

The Euclidian norm for vectors is denoted by $\|\cdot\|$. For functions, we introduce the seminorm $\|\cdot\|_{\mathcal{H}}$:

$$\|\varphi\|_{\mathcal{H}} = \sqrt{\|\varphi(0)\|^2 + \int_{-H}^{0} \|\varphi(\theta)\|^2 d\theta}.$$

The matrix $K(t)$, known as the *fundamental matrix* of system (1), satisfies the equality (Bellman & Cooke, 1963):

$$\dot{K}(t) = \sum_{j=0}^{m} K(t-h_j)A_j + \int_{-H}^{0} K(t+\theta)G(\theta)d\theta$$

for $t \geqslant 0$, and the initial conditions $K(0) = I,\ K(t) = 0, t < 0$.

3. Lyapunov–Krasovskii framework

A functional $v_0(x_t(\varphi))$ with prescribed derivative along the trajectories of system (1), given by

$$\frac{dv_0(x_t(\varphi))}{dt} = -x^T(t,\varphi)Wx(t,\varphi),$$

where W is a positive definite matrix, was introduced in Huang (1989). It has the form

$$\begin{aligned}
v_0(\varphi) &= \varphi^T(0)U(0)\varphi(0) \\
&\quad + 2\varphi^T(0)\sum_{j=1}^{m}\int_{-h_j}^{0} U(-\theta-h_j)A_j\varphi(\theta)d\theta \\
&\quad + \sum_{k=1}^{m}\sum_{j=1}^{m}\int_{-h_k}^{0}\varphi^T(\theta_1)A_k^T \\
&\quad \cdot \int_{-h_j}^{0} U(\theta_1+h_k-\theta_2-h_j)A_j\varphi(\theta_2)d\theta_2 d\theta_1 \\
&\quad + 2\varphi^T(0)\int_{-H}^{0}\int_{-H}^{\theta} U(\xi-\theta)G(\xi)d\xi\varphi(\theta)d\theta \\
&\quad + 2\sum_{j=1}^{m}\int_{-h_j}^{0}\int_{-H}^{0}\int_{-H}^{\theta_2}\varphi^T(\theta_1)A_j^T \\
&\quad \cdot U(\theta_1+h_j-\theta_2+\xi)G(\xi)\varphi(\theta_2)d\xi d\theta_2 d\theta_1 \\
&\quad + \int_{-H}^{0}\int_{-H}^{0}\varphi^T(\theta_1)\int_{-H}^{\theta_1}\int_{-H}^{\theta_2}G^T(\xi_1) \\
&\quad \cdot U(\theta_1-\xi_1-\theta_2+\xi_2)G(\xi_2)d\xi_2 d\xi_1\varphi(\theta_2)d\theta_2 d\theta_1.
\end{aligned}$$

Each term of the sum depends on the matrix-valued function $U(\tau)$, named the *delay Lyapunov matrix* associated to W. It satisfies the following set of equations, called dynamic, symmetry and algebraic properties:

$$U'(\tau) = \sum_{j=0}^{m} U(\tau-h_j)A_j + \int_{-H}^{0} U(\tau+\theta)G(\theta)d\theta, \quad \tau \geqslant 0, \tag{2}$$

$$U(\tau) = U^T(-\tau),$$

$$\begin{aligned}
-W &= \sum_{j=0}^{m}\left[A_j^T U^T(-h_j) + U(-h_j)A_j\right] \\
&\quad + \int_{-H}^{0}\left[G^T(\theta)U^T(\theta) + U(\theta)G(\theta)\right]d\theta.
\end{aligned} \tag{3}$$

It is straightforward to prove that for $\tau < 0$,

$$U'(\tau) = -\sum_{j=0}^{m} A_j^T U(\tau+h_j) - \int_{-H}^{0} G^T(\theta)U(\tau-\theta)d\theta. \tag{4}$$

In Kharitonov (2013), it was shown that the Lyapunov matrix of system (1), associated with a given matrix W, is unique if and only if the *Lyapunov condition* holds, i.e., there are no eigenvalues s_1 and s_2 such that $s_1 + s_2 = 0$. Moreover, the addition to $v_0(\varphi)$ of a quadratic term led to the so-called *complete type functionals*, which admit a quadratic lower bound when the system is exponentially stable, and whose derivative captures the whole state of the delay system.