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Brief paper

Structural properties of networked systems with random communication links ☆

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ABSTRACT

The paper deals with the problem of choosing the communication structure of networked systems and shows that a small percentage of all possible communication links suffices to get short paths from the leader to all followers. It considers agents that are connected in a single row and that choose randomly additional communication links within the region of the next M neighbours. Without additional communication links, the path length increases linearly along the row of the agents, which leads to an unsatisfactory performance of agents at the end of the row. The paper derives an analytical expression for the expected path length in the network with the additional links and shows that a small number of additional links considerably shortens the path length from the leader to all followers.

1. Introduction

The paper is concerned with the question how many communication links within a networked system are necessary to ensure a satisfactory system performance. It considers a set of agents that are connected to a single row as depicted in the top part of Fig. 1. Every agent chooses additional communication links towards some of its M nearest neighbours with the probability p (middle part) and selects from the set of incoming links the edge that starts in the node with the lowest index. A directed spanning tree results as the effective communication structure $\mathcal{G}$ of the networked system (lower part of the figure). The restriction of the additional links to the M nearest neighbours is motivated from the practical experience that in wireless communication each node can be directly connected only to nodes in a restricted distance.

The paper describes three results: (i) It presents an analytical expression for the average length v_i of the path from the leader towards the ith agent (Theorem 2). (ii) It shows that the path length can be shortened monotonously by introducing additional communication links. (iii) It presents quantitative results that imply that a small percentage of $p \approx 10\%$ of all possible communication links and a small neighbourhood of $M \approx 5$ suffice to get short links from the leader towards all follower nodes.

The problem considered is relevant, for example, for the selection of the communication structure of multi-agent systems, where all agents should follow a common reference signal generated by the leader. To reach this goal, the agents have to communicate their output $y_i(t)$ over a data network to some of their neighbours. As shown by Lunze (2013), the performance of the overall system can be directly related to the length of the path from the leader towards any individual follower. The result of this paper shows that a small part of all possible communication links suffices to get short paths from the leader towards all agents and, hence, a satisfactory overall system behaviour.

This paper uses ideas of Network Science (Newman, 2000; Strogatz, 2001; Watts, 2003), which has shown that even in large networks a small number of edges suffices to get short paths between arbitrary pairs of vertices ("small-world property"). This idea is transferred in this paper to networked control systems by using the communication structure selection procedure summarised in Section 2 and by proving that the average path length in the resulting graphs is considerably shortened by the random additional communication links that the individual agents may choose.

The effect of random communication on the performance of networked systems has been investigated in the literature on consensus problems with the work of Hatano and Mesbahi (2005) as one of the earliest publications. As the existence of information channels between pairs of agents at each time instant is probabilistic, the overall system has a time-varying topology. Several restrictions have been imposed on the random selection procedure of the network. For example, Abaid and Porfiri (2011) introduced the

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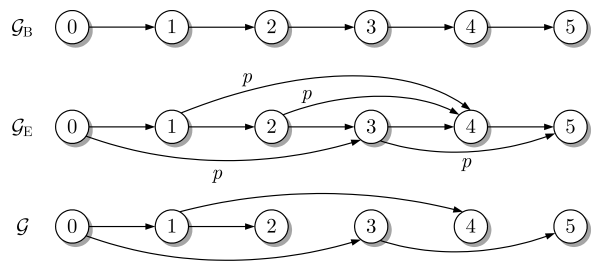

Fig. 1. Communication graphs.

constraint that the number of neighbours is restricted and fixed. For such systems, the main problem considered asks under what conditions on the probability distribution asymptotic consensus appears in the network. This paper also considers random information channels, but the overall network is fixed once it is generated by the procedure described in Section 2. Compared to the cited publications, a different problem arises, namely the problem to evaluate the path length in the resulting network.

Other authors expressed the problem of selecting the communication structure of networked systems as an optimisation problem, e.g. Groß and Stursberg (2011), Gusrialdi and Hirche (2010) and Lunze (2014). Wang, Zhao, Jia, and Guan (2012) extended a given communication structure by additional links in order to ensure the bi-connectivity of the network. Kar and Moura (2008) aimed at a communication structure in which a consensus problem can be solved even if some communication links randomly fail. Massioni and Verhaegen (2009) investigated how the communication structure can be tailored to the structure of the plant. The main difference between this literature and the present paper is the fact that the solutions proposed in literature use a global viewpoint on the overall system, whereas this paper presents a selection method that can be applied by the agents separately based on their local information. The presented method to get the effective communication graph does not require the agents to know the communication links among other agents.

2. Communication structure

2.1. Graph-theoretic notions

The communication structure of the overall system is represented as directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of vertices $\mathcal{V} = \{0, 1, 2, \ldots, N\}$ and the set of edges $\mathcal{E}$. The vertex $0 \in \mathcal{V}$ is associated with the leader agent. A directed edge from the vertex j towards the vertex i is denoted by $(j \to i)$. The graph $\mathcal{G}$ has the adjacency matrix

$$\boldsymbol{K} = (k_{ij}) \quad \text{with} \begin{cases} k_{ij} = 1 & \text{if } (j \to i) \in \mathcal{E} \\ k_{ij} = 0 & \text{else,} \end{cases}$$

where the enumeration of the rows and columns starts with 0. For a path from the leader vertex 0 towards a vertex i, which is denoted by $0 \to i$, the length $|0 \to i|$ is defined to be the number of edges.

2.2. Selection of the communication structure

1. Basic communication graph $\mathcal{G}_B$. This paper considers agents that are connected to a single row and are enumerated accordingly (top part of Fig. 1). This graph is called the basic communication graph with the adjacency matrix $\boldsymbol{K}_B$ with

$$k_{Bij} = \begin{cases} 1 & \text{for } j = i - 1 \\ 0 & \text{else,} \end{cases} \qquad i, j = 0, 1, \ldots, N. \tag{1}$$

2. Extended communication graph $\mathcal{G}_E$. Each agent i chooses additional communication links towards its M nearest neighbours $i + 1, i + 2, \ldots, i + M$ with probability p. The result is called the extended communication graph $\mathcal{G}_E$ (middle part of Fig. 1). The adjacency matrix $\boldsymbol{K}_E$ has the elements k_{Eij} with

$$\text{Prob}\,(k_{Eji} = 1) = p, \quad j = i + 2, i + 3, \ldots, i + M \tag{2}$$

$$\text{Prob}\,(k_{Ei+1,i} = 1) = 1, \tag{3}$$

$(i = 1, 2, \ldots, N)$. The set $\mathcal{P}_i$ of predecessors of the vertex i is given by $\mathcal{P}_i = \{j \in \mathcal{V} \mid k_{Eij} = 1\}$, $(i = 1, 2, \ldots, N)$.

3. Effective communication graph $\mathcal{G}$. Any agent i selects the agent with the smallest index

$$\text{Pred}\,(i) = \min \mathcal{P}_i \tag{4}$$

as predecessor. The edges $(\text{Pred}\,(i) \to i)$, $(i = 1, 2, \ldots, N)$ comprise the effective communication graph $\mathcal{G}$ (lower part of Fig. 1) with the adjacency matrix $\boldsymbol{K} = (k_{ij})$

$$k_{ii} = 0 \quad \text{and} \quad k_{0i} = 0, \quad i = 0, 1, \ldots, N$$

$$k_{ij} = \begin{cases} 1 & \text{if } j = \text{Pred}\,(i) \\ 0 & \text{else} \end{cases} \qquad i = 1, 2, \ldots, N. \tag{5}$$

Obviously, the relations

$$\text{Pred}\,(i) < i, \quad i = 1, 2, \ldots, N \tag{6}$$

$$k_{10} = 1 \tag{7}$$

are valid.

2.3. Tree structure of the effective communication graph

The graph $\mathcal{G}$ has two important properties:

Theorem 1. *$\mathcal{G}$ is a directed spanning tree with the leader vertex* 0 *as the root. Its adjacency matrix $\boldsymbol{K}$ is lower triangular.*

Corollary 1. *For every vertex $i \in \mathcal{V}$ there exists a unique path $0 \to i$ in the effective communication graph $\mathcal{G}$.*

Proof. The property of the matrix $\boldsymbol{K}$ to be lower triangular results directly from Eqs. (5), (7). Furthermore, no cycles exist in the graph. It remains to show that for every vertex i, there exists exactly one path $0 \to i$. This fact will be proved by induction.

For the vertex 1, there is exactly one path $0 \to 1$, because $\boldsymbol{K}$ is lower triangular and Eq. (7) holds. Assume now that for the vertices $i \in \{1, 2, \ldots, k\}$ there exists exactly one path $0 \to i$ in the graph $\mathcal{G}$. As the edge $(\text{Pred}\,(k+1) \to k+1)$ belongs to the graph $\mathcal{G}$ with $\text{Pred}\,(k + 1) \in \{0, 1, \ldots, k\}$, there exists exactly one path $0 \to k + 1$, which either consists of the path $0 \to \text{Pred}\,(k + 1)$ together with the edge $(\text{Pred}\,(k + 1) \to k + 1)$ or of the edge $(0 \to k + 1)$. Hence, the graph $\mathcal{G}$ is a directed spanning tree with the root vertex 0. □

3. Properties of the effective communication graph

3.1. Determination of the average path lengths

This section proves an analytical relation between the average path length v_i, the connection probability p and the number M of neighbours to which each agent can choose a communication link. The expected value v_i of the length of the path $P = 0 \to i$ is given by

$$v_i = \sum_{P \in \mathcal{P}(0 \to i)} \text{Prob}\,(P) \cdot |P| \tag{8}$$

with the sum going over all paths P from the vertex 0 towards the vertex i that may exist in the graph $\mathcal{G}$.

As any path $0 \to i$ consists of the path $0 \to \text{Pred}\,(i)$ and the edge $(\text{Pred}\,(i) \to i)$ with $\text{Pred}\,(i)$ satisfying Eq. (6), v_i can be represented recursively with $v_0 = 0$ and

$$v_i = \sum_{j=0}^{i-1} \text{Prob}\,(\text{Pred}\,(i) = j)\, v_j + 1, \quad i = 1, 2, \ldots, N. \tag{9}$$