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Multiperiod mean-standard-deviation time consistent portfolio selection ☆

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ABSTRACT

We study a multiperiod portfolio selection problem in which a single period mean-standard-deviation criterion is used to construct a separable multiperiod selection criterion. Using this criterion, we obtain a closed form optimal strategy which depends on selection schemes of investor's risk preference. As a consequence, we develop a multiperiod portfolio selection scheme. In doing so, we adapt a pseudo dynamic programming principle from other existing results. The analysis is performed in the market of risky assets only, however, we allow both market transitions and intermediate cash injections and offtakes.

1. Introduction

Portfolio selection problem has been of a great interest by both academics and practitioners. There are various selection criteria available. Some examples include the classical mean–variance (MV) criterion introduced by Markowitz (1952), the safety-first criterion proposed by Roy (1952), and the criterion which targets a particular wealth level used by Skaf and Boyd (2009). In this paper, we choose a mean-standard-deviation (MSD) criterion which (in the single period case) has the form:

$$J_x(\boldsymbol{u}) = \mathbb{E}_x(W^{\boldsymbol{u}}) - \kappa\sqrt{\mathrm{Var}_x(W^{\boldsymbol{u}})},$$

where $W^{\boldsymbol{u}}$ denotes investor's wealth at the end of the investment horizon, which depends on investor's initial wealth x, his investment strategy $\boldsymbol{u}$, and a parameter $\kappa > 0$ which characterizes investor's risk tolerance. All terms will be defined in a more precise way later. There are several reasons to choose this criterion. The most significant one is the fact that it provides a partial understanding on how to choose a dynamic portfolio for the class of translation-invariant and positive-homogeneous (TIPH) risk measures. The TIPH risk measure class contains many interesting examples such as the well-known Value at Risk (VaR), and the Conditional Value at Risk (CVaR). In a single period portfolio selection model, it has been shown (see for example Landsman & Makov, 2011) that if the asset returns follow a (joint) elliptical distribution, optimizing a risk measure from the TIPH class is equivalent to optimizing the MSD criterion.

There has been extensive research in the past regarding single period portfolio selection by using MSD criterion. For example, Landsman (2008) found a closed form solution by using matrix partitions. Owadally (2012) proposed two alternative ways in which the obtained solutions are more efficient computationally. The first approach is based on the relationship between optimizing the MSD criterion and optimizing the MV criterion which is close to a precommitment approach. For portfolio selection by precommitment approach, we refer to Çakmak and Özekici (2006); Li and Ng (2000). The second approach utilizes the standard Lagrange argument together with some facts from linear algebra. One may note that both (Landsman, 2008; Owadally, 2012) consider a market of risky assets. Later on, a risk free asset is added

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to the model in Landsman and Makov (2012), however only a trivial solution is obtained (when a budget constraint only is imposed).

Just like in the second method given by Owadally (2012) we follow a standard Lagrange method to solve the single period problem. However, the main interest of this paper, is to extend the single period framework to a multiperiod model. In doing so, we note that the MSD and the MV criterion face the same difficulty due to the presence of the variance term in their formulation. The difficulty is that we cannot apply the standard dynamic programming principle (DPP). In recent years, it is quite popular to use the time consistency concept to establish a pseudo DPP. This concept has been widely applied in the multiperiod portfolio selection problem with the MV criterion. We mention a few references here: Bensoussan, Wong, Yam, and Yung (2014), Björk and Murgoci (2010) Chen, Li, and Guo (2013) and Wu (2013) and for discrete time, and (Bensoussan et al., 2014; Björk, Murgoci, & Zhou, 2014) for continuous time setting. There are different definitions of time consistency. Here, we concentrate on the time consistency of optimal strategy with respect to a multiperiod selection criterion. To formulate a pseudo DPP, it has been argued that a rational investor should choose his strategy consistently through time. In other words, the investors only choose among strategies which they are going to follow in the future (see Strotz, 1955–1956). Thus, in discrete time, by utilizing this time consistency approach one can select an optimal strategy through a period-wise optimization and backward recursion. A meaningful explanation is given through a game theory point of view, and such a strategy has been called an equilibrium control (henceforth referred to as a weakly time consistent optimal strategy). It inherits the equilibrium concept that arises in game theory. We refer to Bensoussan et al. (2014), Björk and Murgoci (2010) and Wu (2013) and the references therein for more details. With a rather strong form of time consistency as proposed, for example by Kang and Filar (2006), an extra property of a time consistent optimal strategy is required. This property states that any sub-strategy of a weakly time consistent optimal strategy is also optimal for the corresponding subsequent periods. This is essentially satisfied for an optimal strategy that can be obtained through the standard DPP. Inspired by the work of Chen et al. (2013) and Kovacevic and Pflug (2009) constructed a multiperiod separable selection criterion. With respect to this criterion, they proved that the optimal strategy obtained through the pseudo DPP satisfies the extra property of strong time consistency. They obtained a closed form optimal strategy with a multiperiod separable selection criterion of MV type. Later on, their work has been extended by Chen, Li, and Zhao (2014) to allow market transitions.

To the authors' best knowledge, the multiperiod portfolio selection problem in which a MSD type criterion is used as a selection criterion is only briefly mentioned in Kronborg and Steffensen (2015). However, the authors consider a model with two assets only, where one of the assets is supposed to be risk free. Within their setting, a trivial result (a special case of Landsman & Makov, 2012) only is obtained. In essence, the outcome is that if the reward is large enough, it would be advisable to invest as much as possible in the risky asset, whereas when the reward is too little in comparison to the investor's risk tolerance, the strategy is to invest in the risk free asset only. A similar result is obtained in the corresponding continuous time problem (see Kronborg & Steffensen, 2015; Kryger & Steffensen, 2010). Our contribution in this paper is to extend the single period model to a multiperiod selection scheme. We take the single period MSD criterion and formulate a separable multiperiod selection criteria of MSD type (similar to Chen et al., 2013 for the MV case). By applying the aforementioned pseudo DPP, we obtain a closed form optimal strategy. The analysis is performed in a market of risky assets only. However, we allow for market transitions, and also for intermediate cash injections and offtakes. Thus, the wealth process of the investor is no longer self-financing in our setting. As far as we are aware, for multiperiod portfolio selection problem, the only work in which intermediate cash injections and offtakes are allowed and closed form solution is obtained, is by Wu and Li (2012). However unlike our work, the authors consider the multiperiod MV criterion, and follow a precommitment approach.

The paper is organized in the following way. In Section 2, we set up our model. In Section 3, we derive the optimal strategy, obtain the multiperiod portfolio selection scheme, and compute the optimal conditional expectation and conditional variance of the terminal wealth. Numerical illustrations and comparisons are performed in Section 4. Finally, we conclude the paper in Section 5. To prepare for our work, we introduce some notations and conventions. We use a bold letter to distinguish a vector $\boldsymbol{v} \in \mathbb{R}^d$ from a scalar $v \in \mathbb{R}$. All vectors are column vectors. Moreover,

- for any matrix $\boldsymbol{B}$, $\boldsymbol{B}^T$ denotes its transpose, $\bar{\boldsymbol{B}}(i)$ denotes the sum of the elements of its ith row; if $\boldsymbol{B}$ is square, $\boldsymbol{B}^m$ denotes its mth power, where $m \geq 0$, and $\boldsymbol{B}^0 = \boldsymbol{I}$ (identity matrix);
- for any vector $\boldsymbol{v}$, we denote by $\boldsymbol{v}^i$ its ith component, and by $\mathrm{diag}(\boldsymbol{v})$ a diagonal matrix whose diagonal elements $\mathrm{diag}(\boldsymbol{v})_{ii} = \boldsymbol{v}^i$ for all i;
- for any matrix $\boldsymbol{B}$, and a vector $\boldsymbol{v}$, we define $\boldsymbol{B}_{\boldsymbol{v}}$ as the matrix product of $\boldsymbol{B}$ and $\mathrm{diag}(\boldsymbol{v})$;
- for any sequence of matrices $(\boldsymbol{B}_n)_{n>0}$, and $m < l$, we put $\sum_{n=\ell}^{m} \boldsymbol{B}_n = 0$, and $\prod_{n=\ell}^{m} \boldsymbol{B}_n = I$.

2. Problem formulation

2.1. The market and the investor

Consider a market which has a finite number of different states such as "Normal", "Bull" and "Bear". From time to time the market may shift from one state to another. The transitions of the market are captured by a discrete time homogeneous Markov Chain $\{\theta_n, n \geq 0\}$, with a state space $S = \{1, \dots, k\}$, and a transition matrix $\boldsymbol{Q} = (q_{ij})_{k\times k}$. There are $d > 1$ risky assets in the market with random return rates $\boldsymbol{r}_n^1, \dots, \boldsymbol{r}_n^d$ evolving over time interval $[0, N]$. The vector process of return rates $(\boldsymbol{r}_n^1, \dots, \boldsymbol{r}_n^d)^T$ will be denoted by $\boldsymbol{r}_n$ whose dynamics is given by an equation

$$\boldsymbol{r}_{n+1}(\theta_n) = \boldsymbol{m}_n(\theta_n) + \boldsymbol{s}_n(\theta_n)\boldsymbol{\epsilon}_{n+1} \in \mathbb{R}^d, \tag{1}$$

(see for example, Costa and Araujo (2008), for this commonly used model). The process $(\boldsymbol{\epsilon}_n)_{n>0}$ is a sequence of independent identically distributed (i.i.d) d-dimensional zero mean random vectors, with covariance matrix I. The functions $\boldsymbol{m}_n : S \to \mathbb{R}^d$ and $\boldsymbol{s}_n : S \to \mathbb{R}^{d\times d}$ are deterministic for each $n = 0, \dots, N-1$. In what follows it will be sometimes convenient to use the notation $\boldsymbol{r}_n(\theta_n)$ for $\boldsymbol{r}_n$. Then, for a given market state $\theta_n = j$, the ith component $\boldsymbol{r}_{n+1}^i(j)$ of $\boldsymbol{r}_{n+1}(j)$ represents the return of the ith risky asset over time period $[n, n+1]$. Thus, for every one dollar, we obtain

$$\boldsymbol{R}_{n+1}(\theta_n) = \boldsymbol{1} + \boldsymbol{r}_{n+1}(\theta_n), \tag{2}$$

where $\boldsymbol{1} \in \mathbb{R}^d$ is a vector of ones. An investor who has a finite investment horizon $[0, N]$, chooses a strategy at time 0, and adjusts his strategy at times $n = 1, \dots, N-1$. We denote the strategy of the investor as

$$\boldsymbol{u} = (\boldsymbol{u}_0(\theta_0), \dots, \boldsymbol{u}_{N-1}(\theta_{N-1}))^T, \tag{3}$$

where each $\boldsymbol{u}_n : S \to U$ is a deterministic function and

$$U = \{\boldsymbol{u} \in \mathbb{R}^d : \boldsymbol{1}^T\boldsymbol{u} = 1\}. \tag{4}$$

For any given market state $\theta_n = j$, the ith component $\boldsymbol{u}_n^i(j) \in \mathbb{R}$ represents the proportions of wealth allocated by the investor to the ith asset. The set $\mathcal{U}^0$ of all such strategies will be interpreted