Automatica 73 (2016) 237–247

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Optimizing the convergence rate of the quantum consensus: A discrete-time model ☆

Saber Jafarizadeh

School of Electrical & Information Engineering, University of Sydney, NSW, 2006, Australia

ARTICLE INFO

Article history:
Received 7 November 2015
Received in revised form
7 June 2016
Accepted 18 June 2016

Keywords:
Quantum networks
Distributed consensus
Aldous' conjecture
Optimal convergence rate

ABSTRACT

Motivated by the recent advances in the field of quantum computing, quantum systems are modeled and analyzed as networks of decentralized quantum nodes which employ distributed quantum consensus algorithms for coordination. In the literature, both continuous and discrete-time models of the algorithm have been proposed. This paper aims at optimizing the convergence rate of the discrete-time quantum consensus algorithm over a quantum network with N qudits. The induced graphs are categorized in terms of the partitions of integer N by arranging them as the Schreier graphs. It is shown that the original optimization problem reduces to optimizing the Second Largest Eigenvalue Modulus of the weight matrix. Exploiting the Specht module representation of partitions of N, the Aldous' conjecture is generalized to all partitions of integer N implying that the spectral gap of all resultant induced graphs is the same. The spectral radius of the Laplacian is obtained from the feasible least dominant partition in the Hasse diagram of integer N. By analytically addressing the semidefinite programming formulation of the problem, closed-form expressions for the optimal results are provided for a wide range of topologies.

1. Introduction

1.1. Related works and applications

Consensus is essential to the coordinated control of dynamical systems modeled as networks of autonomous agents, such as power grids and social networks (Kocarev, 2013; Ren & Beard, 2008). Reaching consensus as a cooperative collective behaviors in networks of autonomous agents have been studied extensively in the context of distributed control and optimization on networks (Jadbabaie, Lin, & Morse, 2003; Olfati-Saber & Murray, 2004; Tsitsiklis, Bertsekas, & Athans, 1986). Optimizing the discrete-time model of the classical distributed average consensus algorithm has been addressed analytically in Jafarizadeh (2015a) and Jafarizadeh and Jamalipour (2011a,b).

In the recent advances in the field of quantum distributed computing (Broadbent & Tapp, 2008; Buhrman & Rohrig, 2003; Denchev & Pandurangan, 2008), quantum systems are analyzed as networks of Quantum nodes that coordinate and carry out computation without any centralized observation. Authors in Sepulchre, Sarlette, and Rouchon (2010) has generalized the concept of consensus to non-commutative spaces by showing the analogy between the stochastic weight matrices employed in classical consensus algorithm and the representation of quantum channels based on Kraus decomposition. They have shown that the classical consensus is a special case of the consensus algorithm defined over non-commutative spaces. Furthermore, they have presented convergence results for quantum stochastic maps where they have used Birkhoff theorem to analyze the asymptotic convergence of a quantum system to a fully mixed state.

Authors in Mazzarella, Sarlette, and Ticozzi (2013), Mazzarella, Ticozzi, and Sarlette (2013) and Mazzarella, Sarlette, and Ticozzi (2015) employ the discrete-time model of the classical consensus algorithm to address the consensus in quantum networks, as a composite (or multipartite) quantum system with multi-qudits. Furthermore, they reinterpret the quantum consensus algorithm as a symmetrization problem, and they derive the general conditions for convergence. They exploit four different schemes for consensus states that can be generalized to the quantum domain, where the classical consensus can be explained only by the symmetry-based consensus state. Therefore, the quantum consensus state is introduced as the symmetric state which is invariant to all permutations. This state is referred to as the trivial representation as well. In other words, in quantum networks, consensus is achieved when the density matrix for each induced

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor James Lam under the direction of Editor Ian R. Petersen.

E-mail address: saber.jafarizadeh@sydney.edu.au.

http://dx.doi.org/10.1016/j.automatica.2016.07.029
0005-1098/

graph converges to a symmetric state which is invariant to all permutations that map the partition associated to each induced graph to itself. Thus, the quantum consensus algorithm is the consensus algorithm over quantum networks with symmetric state as the consensus state. Note that in the case of quantum networks, the final symmetric state for different induced graphs are not necessarily the same.

Both continuous-time and discrete-time models have been considered for analysis of the consensus algorithm over quantum networks (Jafarizadeh, 2015b; Shi, Dong, Petersen, & Johansson, 2015; Shi, Fu, & Petersen, 2015a,b). In Shi et al. (2015,a,b), authors consider the continuous time model of the classical consensus dynamics where their approach is based on the induced graphs of the quantum interaction graph. They establish necessary and sufficient conditions for exponential and asymptotic quantum consensus, respectively, for switching quantum interaction graphs. In Jafarizadeh (2015b), the convergence rate of the continuous time quantum consensus is optimized and it is shown that the optimal convergence rate is independent of the value of d in qudits. The classical consensus algorithm is a special case of the consensus algorithm over quantum networks. Since in the classical case, the state of all agents converge to the average of the agents' initial states (which is also invariant to all possible permutations). Thus the consensus algorithm over quantum networks is a generalization of the classical one.

Even though the consensus algorithms over quantum networks are in the early stages of their development but these algorithms lay the foundation for development of several other distributed algorithms for applications such as the reduced-state synchronization of qubit networks (Shi, Fu, & Petersen, 2013) and multiplex Markov chain (Trpevski, Stanoev, Koseska, & Kocarev, 2014). In principle, for engineered systems and networks, discrete-time model presents the opportunity for designing algorithmic procedures with finite convergence time while in the continuous-time model the convergence is asymptotic. Also compared to the continuous-time model, the discrete-time model of a system enables the implementation of a more sophisticated controller. The semidefinite formulations along with the optimal results presented in this paper can be directly applied for optimization of such algorithms.

1.2. Main results

In this paper, we optimize the convergence rate of the discrete-time model of the quantum consensus over a quantum network with N qudits. Unlike the results obtained for the continuous time model (Jafarizadeh, 2015b), the convergence rate of the algorithm depends on the value of d in qudits.

First we expand the density matrix in terms of the generalized Gell-Mann matrices. By doing so, we have shown that the discrete-time quantum consensus algorithm is equivalent to a classical discrete-time consensus algorithm where its underlying graph (with Laplacian matrix $\boldsymbol{L}_Q$) is a cluster of connected components where each connected graph component corresponds to a given partition of N (number of qudits in network). The convergence rate of the equivalent classical discrete-time consensus algorithm is directly related to the spectral radius of the weight matrix ($\boldsymbol{W}_Q$) or equivalently the Second Largest Eigenvalue Modulus (SLEM) of $\boldsymbol{W}_Q$, which is dictated by the second smallest ($\lambda_2(\boldsymbol{L}_Q)$) and the greatest ($\lambda_{\max}(\boldsymbol{L}_Q)$) eigenvalues of $\boldsymbol{L}_Q$. We show that the induced graphs are the Schreier graphs and using the Young Tabloids, we sort the induced graphs obtained from all possible partitions of the integer N. Exploiting the Specht module representation of partitions of N, we have shown that the spectrum of the Laplacian corresponding to the less dominant partition in the Hasse Diagram includes that of the one level dominant partition. Therefore the Laplacian matrix corresponding to partition $(1, 1, \dots, 1)$ includes the corresponding spectrum of all other partitions. Based on this result and the Aldous' conjecture (Aldous & Fill, 2014) we have shown that the second smallest eigenvalues ($\lambda_2(\boldsymbol{L}_{(n)})$) of the Laplacian of all partitions (except (N)) in the Hasse diagram are equal. This is the generalization of the Aldous' conjecture to all partitions (except (N)) in the Hasse diagram of integer N.

Applying the generalization of the Aldous' conjecture, we have proved that $\lambda_2(\boldsymbol{L}_Q)$ can be calculated from the Laplacian matrix corresponding to any of the partitions (other than $n = (N)$), where the most suitable one is partition $(N-1, 1)$. Unlike $\lambda_2(\boldsymbol{L}_{(n)})$, the greatest eigenvalue ($\lambda_{\max}(\boldsymbol{L}_{(n)})$) of the induced graphs corresponding to different partitions are not the same. Selecting the appropriate induced graph that contains $\lambda_{\max}(\boldsymbol{L}_Q)$ depends on the value of N and d. We have shown that the problem of optimizing the convergence rate of the discrete-time quantum consensus algorithm is directly related to W (sum of the weights on edges of the underlying graph) and the second smallest ($\lambda_2(\boldsymbol{L}_{(\boldsymbol{N-1,1})})$) and the greatest ($\lambda_{\max}(\boldsymbol{L}_{(\boldsymbol{N-1,1})})$) eigenvalues of the Laplacian matrix corresponding to partition $(N-1, 1)$. For $N \le d^2$, the greatest eigenvalue ($\lambda_{\max}(\boldsymbol{L}_Q)$) is obtained from Laplacian matrix corresponding to partition $(1, 1, \dots, 1)$ which is $2W$ (and W is sum of the weights on edges of the underlying graph). But for larger values of N, partition $(1, 1, \dots, 1)$ is not feasible and the greatest eigenvalue ($\lambda_{\max}(\boldsymbol{L}_Q)$) is included in partitions dominant to $(1, 1, \dots, 1)$. Thus we can deduce that all eigenvalues of the weighted adjacency matrices corresponding to the induced graphs are bounded by W in absolute value.

Based on the above results, it is concluded that for $N \le d^2$, the constraint $W < 1$ is the necessary and sufficient condition for convergence of the quantum consensus algorithm to the consensus point, whereas for $N \ge d^2 + 1$, the constraint $W \le 1$ is the sufficient condition for the convergence of the algorithm. These results are in agreement with that of Mazzarella, Sarlette et al. (2013) showing that completely positive and trace preserving maps guarantee the convergence of the consensus algorithm. Furthermore, for calculating the optimal weights and SLEM, we have provided the semidefinite programming formulation of the problem for three categories, namely for $N \le d^2$, $N = d^2 + 1$ and $N > d^2 + 1$. We have shown that for the last two categories, SLEM is governed by the $\lambda_2(\boldsymbol{L}_Q)$ and the initial optimization problem for these categories reduces to the optimization problem of the continuous-time quantum consensus algorithm with the constraint $W = 1$. In addition, we have analytically addressed the semidefinite programming formulation of the problem for a wide range of topologies and provided closed-form expressions for the optimal convergence rate and the optimal weights.

The rest of the paper is organized as follows. Section 2 presents some preliminaries including relevant concepts in graph theory, discrete-time consensus algorithm, Qubits, Qudits and Quantum Dynamics. Section 3 describes the derivation of the discrete-time quantum consensus algorithm in terms of the quantum gossip interaction proposed in Mazzarella, Sarlette et al. (2013), Mazzarella, Ticozzi et al. (2013), Mazzarella et al. (2015) and explains the transformation of its optimization problem into the optimization of the classical discrete-time consensus algorithm. In Section 4, analytical optimization of the discrete-time consensus problem and closed-form expressions of the optimal results for a range of topologies have been presented. Section 5 concludes the paper.

2. Preliminaries

In this section, we present the fundamental concepts from graph theory and classical Discrete-Time Consensus (DTC) algorithm. Regarding Young tableau, S_N group and its representation,