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Barriers and potentially safe sets in hybrid systems: Pendulum with non-rigid cable[☆]

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ABSTRACT

This paper deals with an application of the notion of barrier in mixed constrained nonlinear systems to an example of a pendulum mounted on a cart with non-rigid cable, whose dynamics may switch to free-fall when the tension of the cable vanishes. We present a direct construction of the boundary of the potentially safe set in which there always exists a control such that the cable never goes slack. A discussion on the dependence of this set with respect to the pendulum and cart masses is then sketched.

1. Introduction

This paper presents an application and slight extension of the recent work on *barriers* in constrained nonlinear systems, see De Dona and Lévine (2013), Esterhuizen (2015), Esterhuizen and Lévine (2015). Given a pendulum on a cart with the rigid bar replaced by a massless cable, we aim at designing a control law which guarantees that the cable always remains taut. The study of this system may be useful to the investigation of safely controlling overhead cranes where slackness of the cable would result in free-fall of the working mass, which would therefore be uncontrolled, and thus potentially harmful for the system and its environment. Such a system whose dynamics may switch conditionally to an event which is, itself, a function of the state and input, is generally called a *hybrid system* (see e.g. Gao, Lygeros, & Quincampoix, 2007; Tomlin, Mitchell, Bayen, & Oishi, 2003; van der Schaft & Schumacher, 2000). The reader may also refer to Kiss (2000), Kiss, Lévine, and Mullhaupt (1999), Kiss, Lévine, and Mullhaupt (2000) for studies on modelling and trajectory planning of *weight handling equipment*. A similar problem appears in Nicotra, Naldi, and Garone (2014) where the authors study *tethered unmanned aerial vehicles* in the different perspective of designing a stabilizing feedback controller.

For a constrained nonlinear control system, the *admissible set* is the set of all initial conditions for which there exists a control such that the constraints are satisfied for all time. Under mild assumptions, this set is closed and its boundary consists of two complementary parts. One of them, called the *barrier*, enjoys the so-called *semi-permeability* property (Isaacs, 1965) and its construction is done via a minimum-like principle (De Dona & Lévine, 2013; Esterhuizen, 2015; Esterhuizen & Lévine, 2015). Our approach to solving the above mentioned problem of the pendulum on a cart is to find this system's admissible set and to guarantee the cable tautness as follows: if the state remains in the admissible set's interior, the control can be arbitrary in some state-dependent constraint set for almost all time and, if the state reaches the barrier, a special control, which we indeed exhibit, needs to be employed in order to keep the cable taut. This admissible set may be interpreted as a *safe set*, or more precisely as a *potentially safe set*.

Note also that we emphasize on systems with *mixed* constraints, i.e. constraints that are functions, in a coupled way, of the control and the state (Clarke & de Pinho, 2010; Esterhuizen & Lévine, 2015; Hestenes, 1966), the reason being that tautness of the cable, which is expressed by the fact that the tension in the cable remains

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nonnegative, can be shown to be equivalent to imposing a mixed constraint. Such constraints are by far more complicated than pure state constraints since they are control dependent, with controls that may be discontinuous with respect to time, thus possibly creating jumps on the constraint set.

Admissible sets are strongly related to invariant sets (Chutinan & Krogh, 2003; Goebel, Sanfelice, & Teel, 2012) and viability kernels (Aubin, 1991; Kaynama, Maidens, Oishi, Mitchell, & Dumont, 2012; Lhommeau, Jaulin, & Hardouin, 2011; Lygeros, Tomlin, & Sastry, 1999; Mitchell, Bayen, & Tomlin, 2005; Tomlin, Lygeros, & Sastry, 2000; Tomlin et al., 2003; van der Schaft & Schumacher, 2000). Our approach contrasts with these works by the fact that in place of computing flows or Lyapunov functions or solutions of Hamilton–Jacobi equations over the whole domain, *we reduce the computations to the boundary of the set under study*. The same kind of comparison also holds with barrier Lyapunov functions (Tee, Ge, & Tay, 2009), or barrier certificates (Prajna, 2006).

The originality of the results of this paper is threefold:

- the interpretation of the cable tautness/slackness as a mixed constraint may be found in Nicotra et al. (2014) but, as already said, with a different stabilization objective. In this paper, we are interested in the analysis and computation of the associated admissible set, namely the largest state domain where one can find an open-loop control such that the cable remains taut, which is new to the authors' knowledge;
- the computation of this admissible set by focusing on its boundary strongly contrasts, in spirit, with the various theoretical constructions found in the literature (Aubin, 1991; Kaynama et al., 2012; Lhommeau et al., 2011; Lygeros et al., 1999; Mitchell et al., 2005; Nicotra et al., 2014; Tomlin et al., 2000, 2003; van der Schaft & Schumacher, 2000) where numerical integration is used to compute flows, each step being simple but the number of steps and iterations exponentially increasing with the dimension of the problem;
- the necessary conditions used here have been obtained in Esterhuizen and Lévine (2015) at the exception of the terminal condition called *ultimate tangentiality condition*. This new terminal condition, introduced to overcome a double problem of singularity and nonsmoothness, is essential for the computation of the barrier: the latter equations cannot be integrated without suitable terminal conditions and the ultimate tangentiality condition of Esterhuizen and Lévine (2015) turned out to be too coarse to obtain a solution.

The paper is organised as follows. In Section 2 we summarize the main results from De Dona and Lévine (2013), Esterhuizen (2015) and Esterhuizen and Lévine (2015) which we present without proofs. In Section 3 we construct the system's barrier. Section 4 provides a discussion of the physical interpretations of the results, and the paper ends with Section 5 that summarizes the conclusions and points out future research.

2. Barriers in nonlinear control system with mixed constraints

2.1. Constrained nonlinear systems with mixed constraints

The contents of this section is borrowed from Esterhuizen (2015) and Esterhuizen and Lévine (2015), where more details may be found. However, Proposition 4 and Theorem 1 of this paper slightly extend the ones of these references. We consider the following nonlinear system with mixed constraints:

$$\dot{x} = f(x, u), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

$$u \in \mathcal{U}, \tag{3}$$

$$g_i\big(x(t), u(t)\big) \le 0 \quad a.e.\ t \in [t_0, \infty) \quad i = 1, \ldots, p \tag{4}$$

where $x(t) \in \mathbb{R}^n$. The set $\mathcal{U}$ is the set of Lebesgue measurable functions from $[t_0, \infty)$ to U, a given compact convex subset of $\mathbb{R}^m$; Thus u is a measurable function such that $u(t) \in U$ for almost all $t \in [t_0, \infty)$.

We denote by $x^{(u,x_0,t_0)}(t)$ the solution of the differential equation (1) at t with input (3) and initial condition (2). Sometimes the initial time or initial condition need not be specified, in which cases we will use the notation $x^{(u,x_0)}(t)$ or $x^u(t)$ respectively.

The constraints (4), called *mixed constraints* (Clarke & de Pinho, 2010; Hestenes, 1966), explicitly depend both on the state and the control. We denote by $g(x, u)$ the vector-valued function whose ith component is $g_i(x, u)$. By $g(x, u) \prec 0$ (resp. $g(x, u) \preceq 0$) we mean $g_i(x, u) < 0$ (resp. $g_i(x, u) \le 0$) for all i. By $g(x, u) \doteq 0$, we mean $g_i(x, u) = 0$ for at least one i. As said before, even if g is smooth, the mapping $t \mapsto g(x(t), u(t))$ is only measurable and the associated mixed constraints are thus assumed to be satisfied almost everywhere.

2.2. The admissible set

We define the following sets:

$$G \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) \preceq 0\} \tag{5}$$

$$G_0 \triangleq \{x \in G : \min_{u \in U} \max_{i \in \{1,\ldots,p\}} g_i(x, u) = 0\} \tag{6}$$

$$G_- \triangleq \bigcup_{u \in U} \{x \in \mathbb{R}^n : g(x, u) \prec 0\}. \tag{7}$$

We further assume:

(**A2.1**) f is an at least C^2 vector field of $\mathbb{R}^n$ for every u in an open subset U_1 of $\mathbb{R}^m$ containing U, whose dependence with respect to u is also at least C^2.

(**A2.2**) There exists a constant $0 < C < +\infty$ such that the following inequality holds true:

$$\sup_{u \in U} |x^T f(x, u)| \le C(1 + \|x\|^2), \quad \text{for all } x$$

where the notation $x^T f(x, u)$ indicates the inner product of the two vectors x and $f(x, u)$.

(**A2.3**) The set $f(x, U)$, called the *vectogram* in Isaacs (1965), is convex for all $x \in \mathbb{R}^n$.

(**A2.4**) g is at least C^2 from $\mathbb{R}^n \times U_1$ to $\mathbb{R}^p$ and convex with respect to u for all $x \in \mathbb{R}^n$.

We also introduce the following state-dependent control set:

$$U(x) \triangleq \{u \in U : g(x, u) \preceq 0\} \quad \forall x \in G. \tag{8}$$

The convexity of U and (A2.4) imply that $U(x)$ is convex for all $x \in G$ and, since g is continuous, the multivalued mapping $x \mapsto U(x)$ is closed with range in the compact set U, and therefore upper semi-continuous (u.s.c.) (see e.g. Berge, 1963; Filippov, 1988).

We assume that, for every $x \in G$, the set $U(x)$ is locally expressible as

$$U(x) \triangleq \{u \in \mathbb{R}^m : \gamma_i(x, u) \le 0, i = 1, \ldots, r\} \tag{9}$$

the functions γ_i being of class C^2, linearly independent, and convex with respect to u for all $x \in G$.

For a pair $(x, u) \in \mathbb{R}^n \times U$, we denote by $\mathbb{I}(x, u)$ the set of indices, possibly empty, corresponding to the "active" mixed constraints:

$$\mathbb{I}(x, u) \triangleq \{i \in \{1, \ldots, r\} : \gamma_i(x, u) = 0\}. \tag{10}$$

The number $\#(\mathbb{I}(x, u))$ of elements of $\mathbb{I}(x, u)$ thus represents the number of "active" constraints among the r independent constraints at (x, u). We further assume: