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Contraction analysis of switched systems via regularization ☆

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ABSTRACT

We study incremental stability and convergence of switched (bimodal) Filippov systems via contraction analysis. In particular, by using results on regularization of switched dynamical systems we derive sufficient conditions for convergence of any two trajectories of the Filippov system between each other within some region of interest. We then apply these conditions to the study of different classes of Filippov systems including piecewise smooth (PWS) systems, piecewise affine (PWA) systems and relay feedback systems. We show that contrary to previous approaches, the our conditions allow the system to be studied in metrics other than the Euclidean norm. The theoretical results are illustrated by numerical simulations on a set of representative examples that confirm their effectiveness and ease of application.

1. Introduction

Incremental stability has been established as a powerful tool to prove convergence in nonlinear dynamical systems (Angeli, 2002). It characterizes asymptotic convergence of trajectories with respect to one another rather than towards some attractor known a priori. Several approaches to derive sufficient conditions for a system to be incrementally stable have been presented in the literature (Angeli, 2002; Forni & Sepulchre, 2014; Lohmiller & Slotine, 1998; Pavlov, van de Wouw, & Nijmeijer, 2006; Russo, di Bernardo, & Sontag, 2010).

A particularly interesting and effective approach to obtain sufficient conditions for incremental stability of nonlinear systems comes from contraction theory (Aminzare & Sontag, 2014; Jouffroy, 2005; Lohmiller & Slotine, 1998). A nonlinear system is said to be *contracting* if initial conditions or temporary state perturbations are forgotten exponentially fast, implying convergence of system trajectories towards each other and consequently towards a steady-state solution which is determined only by the input (the *entrainment* property, e.g. Russo et al., 2010). A vector field can be shown to be contracting over a given K-reachable set by checking the uniform negativity of some matrix measure μ of its Jacobian matrix in that set (Russo et al., 2010). Classical contraction analysis requires the system vector field to be continuously differentiable.

In this paper, we consider an important class of non-differentiable vector fields known as *piecewise smooth* (PWS) *systems* (Filippov, 1988). A PWS system consists of a finite set of ordinary differential equations

$$\dot{x} = f_i(x), \quad x \in S_i \subset \mathbb{R}^n \tag{1}$$

where the smooth vector fields f_i, defined on disjoint open regions S_i, are smoothly extendable to the closure of S_i. The regions S_i are separated by a set Σ of codimension one called the *switching manifold*, which consists of finitely many smooth manifolds intersecting transversely. The union of Σ and all S_i covers the whole state space $U \subseteq \mathbb{R}^n$.

Piecewise smooth systems are of great significance in applications, ranging from problems in mechanics (friction, impact) and biology (genetic regulatory networks) to variable structure systems in control engineering (sliding mode control Utkin, 1992)—for an overview see di Bernardo, Budd, Champneys, and Kowalczyk (2008).

The theoretical study of PWS systems is important. Firstly, the classical notion of solution is challenged in at least two distinct ways. When the normal components of the vector fields either side of Σ are in the *same* direction, the gradient of a trajectory is discontinuous, leading to Carathéodory solutions (Filippov, 1988). In this case, the dynamics is described as *crossing* or *sewing*. But when the normal components of the vector fields on either side of Σ are in the *opposite* direction, a vector field on Σ needs to be defined. The precise choice is not unique and depends on

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the nature of the problem under consideration. One possibility is the use of differential inclusions. Another choice is to adopt the Filippov convention (Filippov, 1988), where a *sliding* vector field f^s is defined on Σ. In this case, the dynamics is described as *sliding*.

Several results have been presented in the literature to extend contraction analysis to non-differentiable vector fields. An extension to piecewise smooth continuous (PWSC) systems was outlined in Lohmiller and Slotine (2000) and formalized in di Bernardo, Liuzza, and Russo (2014). Contracting hybrid systems were analyzed in Lohmiller and Slotine (2000) while the stability analysis of hybrid limit cycles using contraction was presented in Tang and Manchester (2014). An extension of contraction theory, related to the concept of *weak* contraction (Sontag, Margaliot, & Tuller, 2015), to characterize incremental stability of sliding mode solutions of planar Filippov systems was first presented in di Bernardo and Liuzza (2013) and later extended to n-dimensional Filippov systems in di Bernardo and Fiore (2014). Finally, incremental stability properties of piecewise affine (PWA) systems were discussed in Pavlov, Pogromsky, van de Wouw, and Nijmeijer (2007) in terms of *convergence*, a stability property related to contraction theory (Pavlov, Pogromsky, van de Wouw, & Nijmeijer, 2004).

In this paper, we take a different approach to the study of contraction in n-dimensional Filippov systems than the one taken in di Bernardo and Fiore (2014) and di Bernardo and Liuzza (2013). In those papers, the sliding vector field f^s was assumed to be defined everywhere and then the contraction properties of its projection onto the switching manifold was considered (together with a suitable change of coordinates). In the current paper, we adopt a new generic approach which directly uses the vector fields f_i and does not need the explicit computation of the sliding vector field f^s. Our method has a simple geometric meaning and, unlike other methods, can also be applied to nonlinear PWS systems.

Instead of directly analyzing the Filippov system, we first consider a *regularized* version; one where the switching manifold Σ has been replaced by a boundary layer of width 2ε. We choose the regularization method of Sotomayor and Teixeira (1996). We then apply standard contraction theory results to this new system, before taking the limit $\varepsilon \to 0$ in order to recover results that are valid for our Filippov system.

2. Mathematical preliminaries and background

2.1. Matrix measures

Given a real matrix $A \in \mathbb{R}^{n\times n}$ and a norm $|\cdot|$ with its induced matrix norm $\|\cdot\|$, the associated *matrix measure* (also called logarithmic norm Dahlquist, 1958; Lozinskii, 1958; Ström, 1975) is the function $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined as

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h}$$

where I denotes the identity matrix. The following matrix measures associated to the p-norm for $p = 1, 2, \infty$ are often used

$$\mu_1(A) = \max_j \left[a_{jj} + \sum_{i\neq j} |a_{ij}| \right]$$

$$\mu_2(A) = \lambda_{\max}\left(\frac{A + A^T}{2}\right)$$

$$\mu_\infty(A) = \max_i \left[a_{ii} + \sum_{j\neq i} |a_{ij}| \right].$$

The matrix measure μ has the following useful properties (Desoer & Haneda, 1972; Vidyasagar, 2002):

(1) $\mu(I) = 1$, $\mu(-I) = -1$.
(2) If $A = \varnothing$, where $\varnothing$ denotes the zero matrix, then $\mu(A) = 0$.
(3) $-\|A\| \le -\mu(-A) \le Re\,\lambda_i(A) \le \mu(A) \le \|A\|$ for all $i = 1, 2, \dots, n$, where $Re\,\lambda_i(A)$ denotes the real part of the eigenvalue $\lambda_i(A)$ of A.
(4) $\mu(cA) = c\,\mu(A)$ for all $c \ge 0$ (positive homogeneity).
(5) $\mu(A + B) \le \mu(A) + \mu(B)$ (subadditivity).
(6) Given a constant nonsingular matrix Q, the matrix measure $\mu_{Q,i}$ induced by the weighted vector norm $|x|_{Q,i} = |Qx|_i$ is equal to $\mu_i(QAQ^{-1})$.

The following theorem can be proved (Aminzare & Sontag, 2014; Vidyasagar, 1978).

Theorem 1. *There exists a positive definite matrix P such that $PA + A^TP < 0$ if and only if $\mu_{Q,2}(A) < 0$, with $Q = P^{1/2}$.*

We now present results on the properties of matrix measures of rank-1 matrices, since we will need these in the sequel. We believe that Lemma 1 is an original result. For any two vectors $x, y \in \mathbb{R}^n$, $x, y \neq 0$, the matrix $A = xy^T$ has always rank equal to 1. This can be easily proved observing that $xy^T = [y_1x\; y_2x\; \dots\; y_nx]$.

Proposition 1. *For any two vectors $x, y \in \mathbb{R}^n$, $x, y \neq 0$ and for any norm we have that $\mu(xy^T) \ge 0$.*

Proof. The proof follows from property 3 of matrix measures as listed above, that is, for any matrix and any norm $\mu(A) \ge Re\,\lambda_i(A)$, $\forall i$, where $Re\,\lambda_i(A)$ denotes the real part of the eigenvalues $\lambda_i(A)$ of A. Therefore, since a rank-1 matrix has $n-1$ zero eigenvalues its measure cannot be less than zero.

The following important result holds for the measure of rank-1 matrices induced by Euclidean norms.

Lemma 1. *Consider the Euclidean norm $|\cdot|_{Q,2}$, with $Q = P^{1/2}$ and $P = P^T > 0$. For any two vectors $x, y \in \mathbb{R}^n$, $x, y \neq 0$, the following result holds*

$$\mu_{Q,2}(xy^T) = 0 \quad \textit{if and only if } Px = -y,$$

otherwise $\mu_{Q,2}(xy^T) > 0$.

Proof. Firstly we prove that $\mu_2(xy^T) = 0$ if and only if x and y are antiparallel. Indeed, from the definition of μ_2, $\mu_2(xy^T)$ is equal to the maximum eigenvalue of the symmetric part $A_s \equiv (A + A^T)/2$ of the matrix $A = xy^T$. The characteristic polynomial $p_\lambda(A_s)$ of A_s is (Bernstein, 2009, Fact 4.9.16)

$$\begin{aligned} p_\lambda(A_s) &= \lambda^{n-2}\left\{\lambda^2 - x^Ty\lambda - \frac{1}{4}\left[x^Txy^Ty - x^Tyy^Tx\right]\right\} \\ &= \lambda^{n-2}\left\{\lambda^2 - x^Ty\lambda - \frac{1}{4}\left[|x|_2^2|y|_2^2 - (x^Ty)^2\right]\right\}. \end{aligned}$$

This polynomial has always $n-2$ zero roots and (in general) two further real roots. It can be easily seen from Descartes' rule that their signs must be opposite. Therefore, the only possibility for them to be nonpositive is that one must be zero while the other is negative. Using again Descartes' rule, this obviously happens if and only if x and y are antiparallel.

Now, assume that $\mu_{Q,2}(xy^T) = 0$ then, using property 6 of matrix measures, we have $\mu_{Q,2}(xy^T) = \mu_2\left(Qxy^TQ^{-1}\right) = \mu_2\left(Qx(Q^{-1}y)^T\right) = 0$, and, from the result proved above, Qx and $Q^{-1}y$ must be antiparallel, i.e. $Qx = -Q^{-1}y$, or equivalently $Px = -y$.

To prove sufficiency, suppose that $Px = -y$, then $Qx = -Q^{-1}y$ and therefore, using again the result above, we have $\mu_{Q,2}(xy^T) = \mu_2(Qxy^TQ^{-1}) = \mu_2(-Qx(Qx)^T) = 0$.

Note that when x or y (or both) are equal to 0 then by property 2 of matrix measures $\mu(xy^T) = 0$.