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Brief paper

Synchronization under matrix-weighted Laplacian[☆]

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ABSTRACT

Synchronization in a group of linear time-invariant systems is studied where the coupling between each pair of systems is characterized by a different output matrix. Simple methods are proposed to generate a (separate) linear coupling gain for each pair of systems, which ensures that all the solutions converge to a common trajectory.

1. Introduction

Synchronization (consensus) of linear systems with general dynamics (as opposed to first- or second-order integrators) has been thoroughly investigated in the last decade. Early results established the convergence of the solutions of coupled systems to a common trajectory via static linear feedback under the condition that the network topology is fixed (Tuna, 2008, 2009). Later, time-varying topologies were allowed in Yang, Roy, Wan, and Saberi (2011). As the limitations of the static feedback have gradually been overcome, more general results employing dynamic feedback emerged; see, for instance, Li, Duan, Chen, and Huang (2010) and Seo, Shim, and Back (2009) for fixed and (Li, Ren, Liu, & Xie, 2013; Seo, Back, Kim, & Shim, 2012) for time-varying topologies.

All of the above-mentioned works, in fact the majority of the studies on synchronization of dynamical systems, cover the dynamics

$$\dot{x}_i = \sum_{j=1}^{q} a_{ij}(x_j - x_i), \quad i = 1, 2, \ldots, q \tag{1}$$

(where $a_{ij} \in \mathbb{R}_{\geq 0}$ and $x_i \in \mathbb{R}^n$) as a special case of their more general setup. An equivalent representation of these systems reads $\dot{x} = -[L_1 \otimes I_n]x$ where $x = [x_1^T\, x_2^T\, \cdots\, x_q^T]^T$ and $L_1 \in \mathbb{R}^{q\times q}$ is the (weighted) Laplacian matrix (Olfati-Saber & Murray, 2004) whose spectral properties have been proved extremely useful in the analysis and design of multi-agent systems.

A pleasant thing about (1) is that its geometric meaning is clear: "*Each agent moves towards the weighted average of the states of its neighbors*", as stated in Cao, Yu, Ren, and Chen (2013). In fact, in the Euler discretization

$$x_i^+ = x_i + \varepsilon \sum_{j=1}^{q} a_{ij}(x_j - x_i) = \sum_{j=1}^{q} w_{ij} x_j \tag{2}$$

the right hand side becomes *the* weighted average for $\varepsilon > 0$ small enough. There are many ways to define *average* and, qualitatively speaking, what any average attempts to achieve is to compute some sort of *center* of the points considered in the computation. Therefore an excusable and sometimes even useful choice for weighted arithmetic mean is obtained by replacing the scalar weights w_{ij} in (2) by symmetric positive semidefinite matrices $P_{ij} = P_{ij}^T \geq 0$ satisfying $\sum_j P_{ij} = I_n$. This suggests on (1) the modification

$$\dot{x}_i = \sum_{j=1}^{q} Q_{ij}(x_j - x_i)$$

where $Q_{ij} \in \mathbb{R}^{n\times n}$ are symmetric positive semidefinite matrices replacing the scalar weights a_{ij}. (We take $Q_{ii} = 0$.) Whence follows the dynamics $\dot{x} = -Lx$ where

$$L = \begin{bmatrix} \sum_j Q_{1j} & -Q_{12} & \cdots & -Q_{1q} \\ -Q_{21} & \sum_j Q_{2j} & \cdots & -Q_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -Q_{q1} & -Q_{q2} & \cdots & \sum_j Q_{qj} \end{bmatrix}_{qn\times qn} \tag{3}$$

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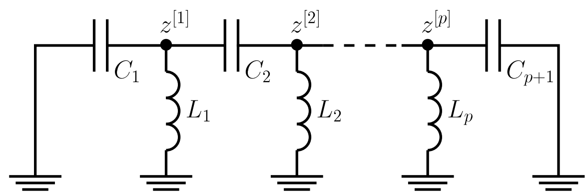

Fig. 1. LC oscillator system.

is the *matrix-weighted Laplacian*. In graph theoretical terms one can say that the graph (with q vertices) associated to this L is such that to each edge a nonzero positive semidefinite matrix Q_{ij} is assigned. Note that for the standard Laplacian the associated graph's edges are assigned weights a_{ij} that are merely positive scalars.

This paper deals with linear time-invariant systems. We consider a synchronization problem where the matrix-weighted Laplacian naturally appears as a tool for both analysis and design. In particular, we study a group of systems whose uncoupled dynamics (described by the matrix A) are identical and the communication between each pair (i, j) of systems has to be realized via a (possibly) different output matrix C_{ij}. Our goal for this setup is to generate linear gains G_{ij} to couple the pairs so that all the solutions in the group converge to a common trajectory. For A neutrally stable, we achieve this goal under detectability (of the pairs (C_{ij}, A) for $C_{ij} \neq 0$) and symmetry ($C_{ij} = C_{ji}$). We also touch the more general situation (where A is allowed to yield unbounded solutions) and establish synchronization under some additional conditions concerning detectability and the strength of connectivity of the network topology. Synchronization in an array where each pair of systems is connected through a different output matrix C_{ij} giving rise to the matrix-weighted Laplacian is yet a relatively unexplored area. Among the very few works employing (in a system-theoretic setting) graphs whose edges are assigned matrix weights is (Barooah, 2008), where the authors study certain relevant applications in distributed estimation.

2. Motivation: coupled LC oscillators

In this section we provide an example array of coupled identical electrical oscillators where the matrix-weighted Laplacian L appears naturally, describing the interconnection of individual systems. For an array of mechanical oscillators, see Tuna (2015).

Consider the individual system in Fig. 1, where p linear inductors ($L_i > 0$) are connected by linear capacitors ($C_i > 0$). The node voltages are denoted by $z^{[i]} \in \mathbb{R}$. Letting $z = [z^{[1]}\, z^{[2]} \cdots z^{[p]}]^T$ the model of this system reads $C\ddot{z} + K^{-1}z = 0$ where $K = \text{diag}(L_1, L_2, \ldots, L_p)$ and

$$C = \begin{bmatrix} C_1 + C_2 & -C_2 & 0 & \cdots & 0 \\ -C_2 & C_2 + C_3 & -C_3 & \cdots & 0 \\ 0 & -C_3 & C_3 + C_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_p + C_{p+1} \end{bmatrix}.$$

Let now an array be formed by coupling q replicas of this system in the arrangement shown in Fig. 2. If we let $z_i \in \mathbb{R}^p$ denote the node voltage vector for the ith system and $g_{ij}^{[k]} = g_{ji}^{[k]} \geq 0$ be the conductance of the resistor connecting the kth nodes of the systems i and j, we can write the dynamics of the coupled systems as $C\ddot{z}_i + K^{-1}z_i + \sum_{j=1}^q G_{ij}(\dot{z}_i - \dot{z}_j) = 0$ where $G_{ij} = \text{diag}(g_{ij}^{[1]}, g_{ij}^{[2]}, \ldots, g_{ij}^{[p]})$. Letting $x_i = [z_i^T\ \dot{z}_i^T]^T$ denote the state of the ith system we at once obtain

$$\dot{x}_i = \begin{bmatrix} 0 & I_p \\ -C^{-1}K^{-1} & 0 \end{bmatrix} x_i + \sum_{j=1}^q \begin{bmatrix} 0 & 0 \\ 0 & C^{-1}G_{ij} \end{bmatrix} (x_j - x_i). \tag{4}$$

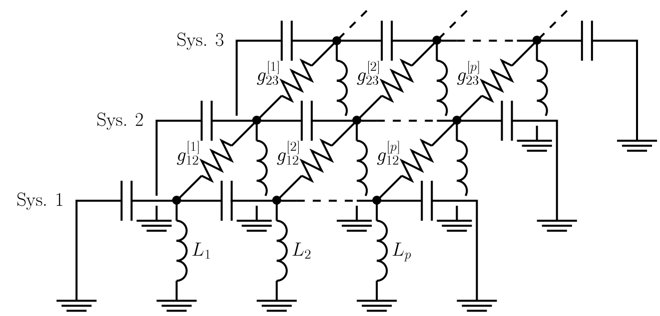

Fig. 2. Array of LC oscillator systems.

Under the coordinate change below

$$\xi_i := \begin{bmatrix} K^{-1/2} & 0 \\ 0 & C^{1/2} \end{bmatrix} x_i$$

we can transform (4) into $\dot{\xi}_i = S\xi_i + \sum_{j=1}^q Q_{ij}(\xi_j - \xi_i)$ where

$$S := \begin{bmatrix} 0 & K^{-1/2}C^{-1/2} \\ -C^{-1/2}K^{-1/2} & 0 \end{bmatrix} \quad \text{and}$$

$$Q_{ij} := \begin{bmatrix} 0 & 0 \\ 0 & C^{-1/2}G_{ij}C^{-1/2} \end{bmatrix}.$$

Note that S is skew-symmetric and $Q_{ji} = Q_{ij} = Q_{ij}^T \geq 0$. Finally, stacking the individual states into a single vector $\xi = [\xi_1^T\ \xi_2^T \cdots \xi_q^T]^T$ the dynamics of the array take the form $\dot{\xi} = ([I_q \otimes S] - L)\xi$, where L is the matrix-weighted Laplacian (3).

3. Problem definition

In this paper we consider a group of linear systems

$$\dot{x}_i = Ax_i + u_i, \quad i = 1, 2, \ldots, q \tag{5a}$$

$$\mathcal{Y}_i = \{C_{i1}(x_1 - x_i), \ldots, C_{iq}(x_q - x_i)\} \tag{5b}$$

with $A \in \mathbb{R}^{n\times n}$, where $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}^n$ is the (control) input of the ith system. The output set $\mathcal{Y}_i$ contains the relative measurements available to the ith system, where $C_{ij} \in \mathbb{R}^{m_{ij}\times n}$ and $C_{ii} = 0$. Associated to the (ordered) set $\{C_{ij}\}$, we let the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represent the network topology, where $\mathcal{V} = \{v_1, v_2, \ldots, v_q\}$ is the set of vertices and a pair (v_j, v_i) belongs to the set of edges $\mathcal{E}$ when $C_{ij} \neq 0$.

The problem we study is the stabilization of the synchronization subspace of the systems (5). In particular, we search for a simple method for choosing the gains $G_{ij} \in \mathbb{R}^{n\times m_{ij}}$ such that under the controls

$$u_i = \sum_{j=1}^q G_{ij}C_{ij}(x_j - x_i) \tag{6}$$

the systems (5) (asymptotically) synchronize. That is, the solutions satisfy $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$ for all indices i, j and all initial conditions. We establish synchronization under two different sets of conditions. We first study the general case where the uncoupled dynamics $\dot{z} = Az$ are allowed to have unbounded solutions and provide certain sufficient conditions for synchronization. Later we will show that if A is neutrally stable, which was the case with the electrical array considered earlier, then synchronization can be achieved under much weaker assumptions.

4. Synchronization under joint Lyapunov detectability

In this section we study synchronization under the assumption below.