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On max-plus linear dynamical system theory: The regulation problem☆

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ABSTRACT

A class of timed discrete event systems can be modeled by using Timed-Event Graphs, a class of timed Petri nets that can have its firing dynamic described by using an algebra called "Max-plus algebra". For this kind of systems it may be desirable to enforce some timing constraints in steady state. In this paper, this problem is called a "max-plus regulation problem". In this context we show a necessary condition for solving these regulation problems and in addition that this condition is sufficient for a large class of problems. The obtained controller is a simple linear static state feedback and can be computed using efficient pseudo-polynomial algorithms. Simulation results will illustrate the applicability of the proposed methodology.

1. Introduction

Timed Event Graphs is an appropriate formalism for modeling some timed discrete event systems, see for instance Atto, Martinez, and Amari (2011), Attia, Amari, and Martinez (2010), Amari, Demongodin, and Loiseau (2004), Kim and Lee (2016), Majdzik, Seybold, and Witczak (2014). These kinds of systems have their dynamics described by linear state-space models in Max-plus Algebra (Baccelli, Cohen, Olsder, & Quadrat, 1992). In some situations it may be desirable that a certain set of constraints in the state space holds. This could be done by using the state variables to design a control law, in analogy with classical control theory.

In the past decade, several papers were published in the problem of synthesizing controllers for this problem when the constraints can be written as max-plus linear equations in the state space (Amari, Demongodin, & Loiseau, 2005; Amari, Demongodin, Loiseau, Jacques, & Martinez, 2012; Atto et al., 2011; Brunsch, Hardouin, & Raisch, 2010; Brunsch, Raisch, & Hardouin, 2012; Gonçalves, Maia, & Hardouin, 2016; Katz, 2007; Maia, Andrade et al., 2011; Maia, Hardouin et al., 2011). See the introduction in Gonçalves et al. (2016) for an in-depth review. In this sense, we highlight the work of Katz (2007) that treated the problem in the light of geometrical control theory, providing sufficient conditions to solve a class of problems. This work was a major inspiration for the developments in our previous work, Gonçalves et al. (2016), and by consequence this one.

Although there is growing interest in the subject, an important feature of controllers was not discussed explicitly until recently: robustness. Indeed, many previous works require that the initial condition lies in a particular set inside the desired specification in order to guarantee that the state will remain on it. But they did not address, at least not explicitly, if it is possible to drive the system from an arbitrary initial condition to the desired specification and then keep it inside this set. This is important because it is closely related to another problem: what would happen if a perturbation – say a machine delays its production – inflicts the system? Would the controller be able to reject this perturbation and return to the desired specification? In other words, we ask for results for the steady-state version of the control problem. As far as the authors knowledge goes, the two only papers that made this discussion explicitly were Kim and Lee (2016) and our previous work, Gonçalves et al. (2016). However, the former only deals with a specific kind of system and specification. Our previous work deals with a general system and a general specification, and we believe that in the mentioned paper we were the first to define and give sufficient conditions to solve this steady-state version of the control problem.

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2. Contributions

This paper builds on our previous work (Gonçalves et al., 2016). In that work, was presented a steady-state version of a max-plus control problem, and two algorithms were derived to solve it in an open-loop strategy. In this paper, we improve the results on one of these algorithms, the *periodic synchronizer*, although the problem that we deal with here is not exactly the same as the one in that paper. In that paper, a larger class of constraints is considered, while here we consider a particular class of these constraints – the ones described by semimodules (see Section 3 for the definition) – which are very common in practice. Indeed, all constraints found in the related papers (Amari et al., 2005, 2012; Atto et al., 2011; Gonçalves et al., 2016; Katz, 2007; Kim & Lee, 2016; Maia, Andrade et al., 2011; Maia, Hardouin et al., 2011) can be rewritten in order to fall into this category. These problems will be denoted hereafter as *max-plus regulation problems*.

The major contribution is that we show that the sufficient conditions derived in Gonçalves et al. (2016) are also *necessary* for solving all max-plus regulation problems under some weak technical assumptions. Additionally, we have shown that the condition also provides a way to solve the problem in *closed-loop* for a wide class of problems. In our previous work the controller acts in open loop and depends on a scalar parameter h. This scalar parameter must be chosen in function of the initial condition and has influence in the upper bound of the number of steps to achieve convergence. Moreover, when the controller firing rate equals the open loop firing rate, the open loop controller may fail unless the parameter h changes dynamically. The closed-loop approach eliminates all these problems: no longer the upper bound in the number of events to converge depends on the initial condition, only in the number of states, and the (closed-loop) approach should work even when the closed-loop radius is equal to the open-loop one, without the necessity of changing any parameter. Indeed, the only parameter is a matrix F, since the control law is a simple static max-plus linear state feedback of the form $u[k] = Fx[k]$ for a constant matrix F.

In order to characterize the class of problems for which the derived condition is necessary and sufficient, the concept of *criticality* is also introduced in this paper. This is related to the *spectrum* of the problem, another concept introduced in this paper. The spectrum is the set of all steady-state possible firing rates under control. In a nutshell, the problem is noncritical – and thus *easy* to solve – if the closed-loop controller that solves the problem is able to delay, even if a little bit, the system in comparison with its open loop behavior. On the other hand, if the problem is critical it may or may not be solved by our methodology. We discuss this topic as well in this paper. Thus, we believe that this paper presents a contribution towards a "final solution" for the regulation problem, that is, a necessary and sufficient condition *for all* problems.

3. Basic definitions

A *Timed-Event Graph* is a subclass of timed Petri nets in which all places have one input and one output transitions. *Max-Plus algebra* is the dioid (idempotent semiring)

$$\mathbb{Z}_{max} = (\mathbb{Z} \cup \{-\infty\}, \oplus, \otimes)$$

in which $\oplus$ is the maximum and $\otimes$ is the traditional sum. More recently, it has been also called *Tropical Algebra*. The symbol $\otimes$ will be frequently omitted and so it will be interpreted by juxtaposition, just like the traditional product in the traditional algebra. So ab reads as $a \otimes b = a + b$. We denote the element $-\infty$ by the symbol ε, and it will also be occasionally denoted by the "null" element. There is also a matricial analogue of this algebra, and so for two matrices A, B of appropriate dimension $A \oplus B$ and $A \otimes B$ will be interpreted as the matricial sum and product with $+$ being replaced by $\oplus$ and $\times$ by $\otimes$. An element in this algebra that has n rows and m columns will be denoted by $\mathbb{Z}_{max}^{n\times m}$, while an element with m rows and one column $\mathbb{Z}_{max}^{m}$. All vectors are column vectors. The symbol A^T denotes the transpose of the matrix A. A vector or matrix of appropriate dimension composed only of ε will also be denoted by ε. The symbol I will denote the max-plus identity matrix of an appropriate order, that is, a matrix in which the diagonal elements are 0 and the others ε. For a natural number k, the kth matrix power A^k will be defined recursively as $A^{k+1} = A^kA$ and $A^0 = I$. If λ is a scalar not ε then $\lambda^{-1} = -\lambda$.

The *Kleene closure* of a square matrix A, denoted by A^*, is equal to $\bigoplus_{i=0}^{\infty} A^i$. The *spectral radius* of this matrix, $\rho(A)$, is the greatest scalar λ for which there exists a vector $v \neq \varepsilon$ in which $Av = \lambda v$, that is, the value $\rho(A)$ is the greatest eigenvalue and v the eigenvector. Generally, even though the entries of the matrix A lie in $\mathbb{Z}$ or are ε, the spectral radius can be a rational number. However, since the units of the problem can be redefined, the entries of the matrix – and thus the spectral radius – can be re-scaled so the spectral radius is either an integer or ε. Thus hereafter we can assume without loss of generality that $\rho(A) \in \mathbb{Z}_{max}$.

A *semimodule*, over a given dioid, is analogous of vector spaces over semirings, that is, a set of elements x together with a scaling $(\lambda, x) \mapsto \lambda x$ and sum $(x, y) \mapsto x \oplus y$ operations which preserve some properties in the context of this given dioid. See Katz (2007) for the formal definition. Finally, *Im M*, the *image* of M, is the semimodule generated by the max-plus column span of the matrix M, that is, if $M \in \mathbb{Z}_{max}^{n\times m}$ then $Im\, M = \{Mv \mid v \in \mathbb{Z}_{max}^{m}\}$.

4. Regulation problem

4.1. Problem statement

Consider a max-plus linear event-invariant dynamical system

$$x[k+1] = Ax[k] \oplus Bu[k], \ \ k \in \mathbb{N};$$
$$x[0] = x_0 \tag{1}$$

for $x[k] \in \mathbb{Z}_{max}^{n}$, $u[k] \in \mathbb{Z}_{max}^{m}$, $A \in \mathbb{Z}_{max}^{n\times n}$ and $B \in \mathbb{Z}_{max}^{n\times m}$. It is *max-plus linear* because its equations can be written in a linear way using the max-plus operators $\oplus$ and $\otimes$, the latter omitted by juxtaposition. It is *event-invariant* because the matrices A and B do not depend on the event k.

It will be assumed without loss of generality that B has no null column (a column full of ε entries). Otherwise, the corresponding control actions would play no role in the system and can be removed.

The *regulation problem*, henceforth denoted by $\mathcal{R}$ (or $\mathcal{R}(A, B, E, D)$ when it is convenient to explicit the matrices), can be defined as follows: find a map $f : \mathbb{Z}_{max}^{n} \times \mathbb{N} \to \mathbb{Z}_{max}^{m}$ such that if $u[k] = f(x[k], k)$ is taken in (1), then there exists a $p \in \mathbb{N}$ such that *for all* initial conditions $x[0]$ we have that *for all* $k \geq p$

$$Ex[k] = Dx[k].$$

The set of $x \in \mathbb{Z}_{max}^{n}$ such that $Ex = Dx$ will be denoted by $\mathcal{S}_{ref}(\mathcal{R})$, the *specification (reference) set*, and is clearly a semimodule. In other words, it is desired to design a (possibly event-varying) state feedback law that leads the dynamical system to a specification set in a finite number of events and keeps it there thereafter, in steady-state, whichever is the initial condition.

Note that, according to the formulation of the problem, it is possible to impose constraints only in *steady state*. If it is strictly necessary that the constraints must hold for *all* $k \geq 0$ then this technique cannot be employed. We refer to Katz (2007) for techniques in this case. Moreover, in order for the constraints to