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Brief paper

Stabilising model predictive control for discrete-time fractional-order systems[☆]

Pantelis Sopasakis[a,1], Haralambos Sarimveis[b]

[a] IMT Institute for Advanced Studies Lucca, Piazza San Fransesco 19, 55100 Lucca, Italy

[b] School of Chemical Engineering, National Technical University of Athens, 9 Heroon Polytechneiou Street, 15780 Zografou Campus, Athens, Greece

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ABSTRACT

In this paper, a model predictive control scheme is proposed for constrained fractional-order discrete-time systems. We prove that constraints are satisfied and we prescribe conditions for the origin to be an asymptotically stable equilibrium point of the controlled system. A finite-dimensional approximation of the original infinite-dimensional dynamics is employed for which the approximation error can become arbitrarily small. The approximate dynamics is used to design a tube-based model predictive controller which steers the system state to a neighbourhood of the origin of controlled size. Stability conditions are finally derived for the MPC-controlled system which are computationally tractable and account for the infinite dimensional nature of the fractional-order system and the state and input constraints. The proposed control methodology guarantees asymptotic stability of the discrete-time fractional order system, satisfaction of the prescribed constraints and recursive feasibility.

1. Introduction

1.1. Background and motivation

Derivatives and integrals of non-integer order, often referred to as *fractional*, are natural extensions of the standard integer-order ones which enjoy certain favourable properties: they are linear operators, preserve analyticity, and have the semigroup property (Hilfer, 2000; Podlubny, 1999). Nonetheless, fractional derivatives are non-local operators, that is, unlike integer-order ones, they cannot be evaluated at a given point by mere knowledge of the function in a neighbourhood of this point and for that reason they are suitable for describing phenomena with infinite memory (Podlubny, 1999).

Fractional dynamics seems to be omnipresent in nature. Examples of fractional systems include, but are not limited to, semi-infinite transmission lines with losses (Clarke, Narahari Achar, & Hanneken, 2004), viscoelastic polymers (Hilfer, 2000), anomalous diffusion in semi-infinite bodies (Guo, Li, & Wang, 2015) and biomedical applications (Magin, 2010) for which Magin et al. provided a thorough review (Magin, Ortigueira, Podlubny, & Trujillo, 2011).

A shift towards fractional-order dynamics in the field of pharmacokinetics may be observed after the classical *in-vitro–in-vivo correlations* theory proved to have faced its limitations (Kytariolos, Dokoumetzidis, & Macheras, 2010). Non-linearities, anomalous diffusion, deep tissue trapping, diffusion across capillaries, synergistic and competitive action and other phenomena give rise to fractional-order pharmacokinetics (Dokoumetzidis & Macheras, 2008). In fact, Pereira derived fractional-order diffusion laws for media of fractal geometry (Pereira, 2010). Increasing attention has been drawn on modelling and control of such systems (Dokoumetzidis & Macheras, 2011; Dokoumetzidis, Magin, & Macheras, 2010; Sopasakis & Sarimveis, 2014), especially in presence of state and input constraints.

Model predictive control (MPC) is an advanced, successful and well recognised control methodology and its adaptation to fractional systems is of particular interest. The current model predictive control framework for fractional-order systems has been developed in a series of papers where integer-order

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E-mail addresses: p.sopasakis@imtlucca.it (P. Sopasakis), hsarimv@central.ntua.gr (H. Sarimveis).

[1] Fax: +39 0583 4326565.

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approximations are used to formulate the control problem (Boudjehem & Boudjehem, 2010; Deng et al., 2010; Romero, de Madrid, Mañoso, & Berlinches, 2010; Romero, de Madrid, Mañoso, Milanés, & Vinagre, 2013). CARIMA (controlled auto-regressive moving average) models are often used in predictive control formulations for the approximation of the fractional dynamics (Joshi, Vyawahare, & Patil, 2014; Romero et al., 2010, 2013). The CARIMA-based approach has been used in various applications such as the heating control of a semi-infinite rod (Rhouma & Bouani, 2014), the power regulation of a solid oxide fuel cell (Deng et al., 2010) and various applications in automotive technology (Romero, de Madrid, Mañoso, & Vinagre, 2012). The well-known Oustaloup approximation has also been used in MPC settings (Romero et al., 2013). It should, however, be noted that such approximations aim at capturing the system dynamics in a range of operating frequencies and, as a result, are not suitable for a rigorous analysis and design of controllers for constrained systems. Additionally, all of the aforementioned works provide examples of unconstrained systems; this shortcoming was in fact identified in the recent paper (Joshi et al., 2014).

Nevertheless, this profusion of purportedly successful paradigms of MPC for fractional-order systems is not accompanied by a proper stability analysis especially when input and state constraints are present. A common denominator of all approaches in the literature is that they approximate the actual fractional dynamics by integer-order dynamics and design controllers for the approximate system using standard techniques. No stability and constraint satisfaction guarantees can be deduced for the original fractional-order system. Currently, one of the very few works on constrained control for fractional-order systems is due to Mesquine et al. where, however, only input constraints are taken into account for the design of a linear feedback controller (Mesquine, Hmamed, Benhayoun, Benzaouia, & Tadeo, 2015).

Hitherto, two approaches can be found in the literature in regard to the stability analysis of discrete-time fractional systems. The first one considers the stability of a finite-dimensional linear time-invariant (LTI) system, known as *practical stability*, but fails to provide conditions for the actual fractional-order system to be (asymptotically) stable (Busłowicz & Kaczorek, 2009; Guerman, Djennoune, & Bettayeb, 2012). This approach is tacitly pursued in many applied papers where stability is established only for a finite-dimensional approximation of the fractional-order system (Romero et al., 2013; Romero, Tejado, Suárez, Vinagre, & de Madrid, 2009). On the other hand, fractional systems can be treated as infinite-dimensional systems for which various stability conditions can be derived (see for example Guermah, Djennoune, & Bettayeb, 2010, Thm. 2), but conditions are difficult to verify in practice let alone to use for the design of model predictive – or other – controllers.

1.2. Contribution

In this paper we describe a stabilising MPC framework for fractional-order systems (of the Grünwald–Letnikov type) subject to state and input constraints. We discretise linear continuous-time fractional dynamics using the Grünwald–Letnikov scheme which leads to infinite-dimensional linear systems. Using a finite-dimensional approximation we arrive at a linear time-invariant system with an additive uncertainty term which casts the discrepancy to the infinite-dimensional system. We then introduce a tube-based MPC control scheme which is known to steer the state to a neighbourhood of the origin which can become arbitrarily small as the order of the approximation of the fractional-order system increases. In our analysis, we consider both state and input constraints which we show that are respected by the MPC-controlled system. We finally prove that under a certain contraction-type condition the origin is an asymptotically stable equilibrium point for the MPC-controlled fractional-order system (see Section 3.2). In this work we provide, for the first time, asymptotic stability conditions (Theorem 4) and we propose a control methodology which guarantees the satisfaction of the prescribed state and input constraints.

This paper builds upon Sopasakis, Ntouskas, and Sarimveis (2015) where the unmodelled part of the system dynamics was cast as a bounded additive uncertainty term and used existing MPC theory to drive the system state in a neighbourhood of the origin without, however, providing any (asymptotic) stability conditions for the origin.

1.3. Mathematical preliminaries

The following definitions and notation will be used throughout the rest of this paper. Let $\mathbb{N}$, $\mathbb{R}^n$, $\mathbb{R}_+$, $\mathbb{R}^{m\times n}$ denote the set of non-negative integers, the set of column real vectors of length n, the set of non-negative numbers and the set of m-by-n real matrices respectively. For any nonnegative integers $k_1 \leq k_2$ the finite set $\{k_1,\ldots,k_2\}$ is denoted by $\mathbb{N}_{[k_1,k_2]}$. Let x be a sequence of real vectors of $\mathbb{R}^n$. The kth vector of the sequence is denoted by x_k and its ith element is denoted by $x_{k,i}$. We denote by $\mathcal{B}^n_\epsilon = \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$ the *open ball* of $\mathbb{R}^n$ with radius ϵ and we use the shorthand $\mathcal{B}^n = \mathcal{B}^n_1$. We define the *point-to-set distance* of a point $z \in X$ from A as $\operatorname{dist}(z, A) = \inf_{a\in A}\|z-a\|$. The space of bounded real sequences is denoted by ℓ^∞. We define the space ℓ^∞_n of all sequences of real n-vectors z so that $(z_{k,i})_k \in \ell^\infty$ for $i \in \mathbb{N}_{[1,n]}$.

Let E be a topological real vector space and $A, B \subseteq E$. For $\lambda \in \mathbb{R}$ we define the scalar product $\lambda C = \{\lambda c : c \in C\}$ and the *Minkowski sum* $A \oplus B = \{a + b : a \in A, b \in B\}$. The Minkowski sum of a finite family of sets $\{A_i\}_{i=1}^k$ will be denoted by $\bigoplus_{i=1}^k A_i$. The Minkowski sum of a sequence of sets $\{A_i\}_{i\in\mathbb{N}}$ is denoted by $\bigoplus_{i\in\mathbb{N}} A_i$ or $\bigoplus_{i=0}^\infty A_i$ and is defined as the *Painlevé–Kuratowski limit* of $\bigoplus_{i=1}^k A_i$ as $k \to \infty$ (Rockafellar & Wets, 1998). The *Pontryagin difference* between two sets $A, B \subseteq E$ is defined as $A \ominus B = \{a \in A : a+b \in A, \forall b \in B\}$. A set C is called *balanced* if for every $x \in C$, $-x \in C$.

2. Fractional-order systems

2.1. Discrete-time fractional-order systems

Let $x : \mathbb{R} \to \mathbb{R}^n$ be a uniformly bounded function, *i.e.*, there is a $M > 0$ so that $\|x(t)\| \leq M$ for all $t \in \mathbb{R}$. The Grünwald–Letnikov fractional-order difference of x of order $\alpha > 0$ and step size $h > 0$ at t is defined as the linear operator (Rhouma, Bouzouita, & Bouani, 2014) $\Delta^\alpha_h : \ell^\infty_n \to \ell^\infty_n$:

$$\Delta^\alpha_h x(t) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} x(t - jh), \tag{1}$$

where $\binom{\alpha}{0} = 1$ and for $j \in \mathbb{N}, j > 0$

$$\binom{\alpha}{j} = \prod_{i=0}^{j-1} \frac{\alpha - i}{i+1} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - j + 1)j!}. \tag{2}$$

The *forward-shifted* counterpart of Δ^α_h is defined as ${}_F\Delta^\alpha_h x(t) = \Delta^\alpha_h x(t+h)$. Now, define

$$c^\alpha_j = (-1)^j \binom{\alpha}{j} = \binom{j-\alpha-1}{j}, \tag{3}$$

and note for all $j \in \mathbb{N}$ that $|c^\alpha_j| \leq \alpha^j/j!$, thus, the sequence $(c^\alpha_j)_j$ is absolutely summable and, because of the uniform boundedness of x, the series in (1) converges, therefore, Δ^α_h is well-defined. It is