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Brief paper

Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks[☆]

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ABSTRACT

The role of restorative coupling on synchronization of coupled identical harmonic oscillators is studied. Necessary and sufficient conditions, under which the individual systems' solutions converge to a common trajectory, are presented. Through simple physical examples, the meaning and limitations of the theorems are expounded. Also, to demonstrate their versatility, the results are extended to cover LTI passive electrical networks. One of the extensions generalizes the well-known link between the asymptotic stability of the synchronization subspace and the second smallest eigenvalue of the Laplacian matrix.

1. Introduction

Studying the collective behavior of coupled harmonic oscillators has been a rewarding enterprize for researchers who try to enhance their understanding on a much-encountered phenomenon in nature: synchronization. For instance, it has been observed that two or more identical pendulums[2] connected by means of dampers eventually swing in unison even if initially they are not synchronized; see Fig. 1. This outcome is not difficult to reach by intuition. Since the energy of the system can only leak out through the dampers, the pendulums should eventually settle to a constant energy state where there is no leakage. No leakage implies that the relative velocities are all zero. In other words, all the pendulums are moving at equal velocities at all times. This is only possible when they are synchronized.

The simple example above has served as a starting point for many significant generalizations. In Ren (2008) Ren studies synchronization of coupled harmonic oscillators allowing time-varying oscillator dynamics as well as time-varying and asymmetrical dampers. The case where the damping between a pair of oscillators becomes effective only when the two are close enough is investigated in Su, Wang, and Lin (2009). The effect of nonlinear damping is analyzed in Cai and Tuna (2010) and of impulsive damping in Zhou, Zhang, Xiang, and Wu (2012). A sampled-data approach is adopted in Sun, Lu, Chen, and Yu (2014) and Zhang and Zhou (2012). Adaptive damping is covered in Su, Chen, Wang, Wang, and Valeyev (2013) and synchronization in the presence of noisy damping is considered in Sun, Yu, Lu, and Chen (2015). Note that all these works consider only dissipative coupling (e.g. dampers). From an engineering point of view this choice is not surprising because introducing restorative coupling (e.g. springs) will in general deteriorate performance by causing longer and more oscillatory transient behavior; for instance, simulation results show that the three pendulums in Fig. 2 synchronize much less rapidly than those in Fig. 1. Perhaps this may partly explain why collective behavior of spring-coupled oscillators has attracted more physicists than engineers. While for the engineer a spring is an option to couple two units, for the physicist it represents an inherent characteristic of interaction. Relevant investigations in the physics community go as far back, if not further, as the work of Fermi, Pasta, and Ulam (1955) where chains of nonlinearly coupled oscillator-like particles were studied. Due to the richness of the subject and the increasing variety of applications in both inanimate and biological systems, the area has maintained its livelihood throughout many decades. See, for instance, Adato, Artar, Erramilli, and Altug (2013), Kapitaniak and Kurths (2014), Kapitaniak, Kuzma, Wojewoda, Czolczynski, and Maistrenko (2014) and Marcheggiani, Chacon, and Lenci (2014) for recent progress.

Through this paper we aim to provide a comprehensive analysis of the collective behavior of identical harmonic oscillators coupled by both restorative and dissipative components. To the best of our

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[2] We restrict our attention to the small oscillations, where the pendulum can be represented by a linear model.

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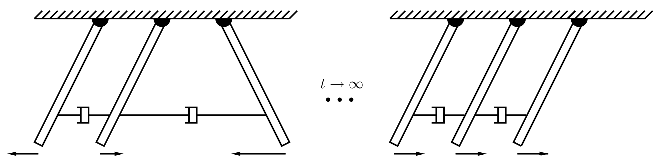

Fig. 1. Damper-coupled pendulums.

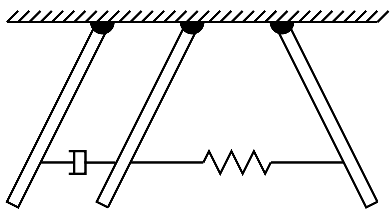

Fig. 2. Damper- and spring-coupled pendulums.

knowledge a detailed treatment of synchronization has not yet been reported for this setting, where two different interconnection graphs are simultaneously at work: the graph representing restorative coupling and the graph representing dissipative coupling. We present a necessary and sufficient condition on the associated pair of Laplacian matrices, under which the individual systems tend to oscillate in unison. We also point out a certain sufficient-only, yet easier-to-check set of conditions guaranteeing synchronization and exercise them on some simple real-world examples for clarity. Later, we attempt to extend our approach to the analysis of linear electrical networks of identical oscillators (of arbitrary order) coupled through passive impedances. For such networks we establish a link between synchronization and the eigenvalues of the (complex) node admittance matrix. This seems to be a natural extension of the well-known connectivity condition in terms of the second smallest eigenvalue of the (real-valued) Laplacian matrix.

2. Coupled harmonic oscillators

Consider the array of q coupled harmonic oscillators

$$\ddot{z}_i + \omega_0^2 z_i + \sum_{j=1}^{q} d_{ij}(\dot{z}_i - \dot{z}_j) + \sum_{j=1}^{q} r_{ij}(z_i - z_j) = 0 \tag{1}$$

($i = 1, 2, \ldots, q$) where $z_i \in \mathbb{R}$ and $\omega_0 > 0$ is the frequency of uncoupled oscillations. The symmetric weights $d_{ij} = d_{ji} \geq 0$ and $r_{ij} = r_{ji} \geq 0$ respectively represent the dissipative and restorative coupling between the ith and jth oscillators. Note that without symmetry, i.e., either $d_{ij} \neq d_{ji}$ or $r_{ij} \neq r_{ji}$, the solutions are not guaranteed to be bounded unless some extra assumption is made. We take $d_{ii} = 0$ and $r_{ii} = 0$. In this section and next we search for conditions on the triple $(\omega_0, (d_{ij})_{i,j=1}^q, (r_{ij})_{i,j=1}^q)$ under which the harmonic oscillators (1) synchronize, i.e., $|z_i(t) - z_j(t)| \to 0$ as $t \to \infty$ for all i, j and all initial conditions.

Let $D, R \in \mathbb{R}^{q\times q}$ denote the weighted Laplacian matrices associated to the topologies described by the dissipative coupling $(d_{ij})_{i,j=1}^q$ and the restorative coupling $(r_{ij})_{i,j=1}^q$, respectively. That is,

$$D = \begin{bmatrix} \sum_j d_{1j} & -d_{12} & \cdots & -d_{1q} \\ -d_{21} & \sum_j d_{2j} & \cdots & -d_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{q1} & -d_{q2} & \cdots & \sum_j d_{qj} \end{bmatrix}$$

and R is constructed similarly. Note that these matrices are symmetric positive semidefinite since $d_{ij} = d_{ji} \geq 0$ and $r_{ij} = r_{ji} \geq 0$. In particular, we can write $z^T D z = \sum_{j>i} d_{ij}(z_i - z_j)^2$ and $z^T R z = \sum_{j>i} r_{ij}(z_i - z_j)^2$, where $z = [z_1\, z_2\, \cdots\, z_q]^T \in \mathbb{R}^q$. Let us now rewrite (1) as

$$\ddot{z} + \omega_0^2 z + D\dot{z} + Rz = 0.$$

This, using $x = [z^T\, \dot{z}^T]^T \in \mathbb{R}^{2q}$, allows us to obtain

$$\dot{x} = \begin{bmatrix} 0 & I_q \\ -(\omega_0^2 I_q + R) & -D \end{bmatrix} x =: \Phi x \tag{2}$$

where $I_q \in \mathbb{R}^{q\times q}$ is the identity matrix. Employing the symmetric positive definite matrix

$$P = \frac{1}{2}\begin{bmatrix} \omega_0^2 I_q + R & 0 \\ 0 & I_q \end{bmatrix}$$

we can establish the following Lyapunov equality

$$\Phi^T P + P\Phi = -\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

Since the right hand side is negative semidefinite, each solution $x(t)$ of the system (2) is bounded. Moreover, by Krasovskii–LaSalle principle, $x(t)$ should converge to the largest invariant region contained in the intersection $\mathcal{D} \cap \{x : x^T P x \leq x(0)^T P x(0)\}$ where

$$\mathcal{D} := \left\{ x : \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} x = 0 \right\}.$$

It turns out that the condition

$$\text{null} \begin{bmatrix} R - \lambda I_q \\ D \end{bmatrix} \subset \text{range } \mathbf{1}_q \quad \text{for all } \lambda \in \mathbb{C} \tag{3}$$

(where $\mathbf{1}_q \in \mathbb{R}^q$ is the vector of all ones) guarantees that this largest invariant region is contained in the synchronization subspace

$$\mathcal{S} := \text{range} \begin{bmatrix} \mathbf{1}_q & 0 \\ 0 & \mathbf{1}_q \end{bmatrix}.$$

In other words:

Lemma 1. *Let (3) hold. Then and only then*

$$x(t) \in \mathcal{D} \quad \text{for all } t \Longrightarrow x(t) \in \mathcal{S} \text{ for all } t \tag{4}$$

where $x(t)$ is the solution of the system (2).

Proof. We first establish (3) $\Longrightarrow$ (4). Let $x(t) = [z(t)^T\, \dot{z}(t)^T]^T$ be a solution of the system (2) that identically belongs to $\mathcal{D}$. This means $D\dot{z}(t) \equiv 0$. Also,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I_q \\ -(\omega_0^2 I_q + R) & 0 \end{bmatrix} x - \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} x \\ &= \begin{bmatrix} 0 & I_q \\ -(\omega_0^2 I_q + R) & 0 \end{bmatrix} x \end{aligned}$$

which implies

$$\ddot{z} + (\omega_0^2 I_q + R) z = 0. \tag{5}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the distinct ($p \leq q$) eigenvalues of R. Since R is symmetric positive semidefinite, these eigenvalues are real and nonnegative. Consequently, the matrix $[\omega_0^2 I_q + R]$ is symmetric positive definite with eigenvalues $\omega_0^2 + \lambda_1, \omega_0^2 + \lambda_2, \ldots, \omega_0^2 + \lambda_p$. Therefore (5) implies that the solution has the form Arnold (1989, Section 23)

$$z(t) = \text{Re} \sum_{k=1}^{p} e^{j\omega_k t} \xi_k \tag{6}$$