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Brief paper

Asymptotic behaviour in the robot rendezvous problem[☆]

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ABSTRACT

This paper presents a natural extension of the results obtained by Feintuch and Francis in (2012a,b) concerning the so-called *robot rendezvous problem*. In particular, we revisit a known necessary and sufficient condition for convergence of the solution in terms of Cesàro convergence of the translates $S^k x_0$, $k \geq 0$, of the sequence x_0 of initial positions under the right-shift operator S, thus shedding new light on questions left open in Feintuch and Francis (2012a,b). We then present a new proof showing that a certain stronger ergodic condition on x_0 ensures that the corresponding solution converges to its limit at the optimal rate $O(t^{-1/2})$ as $t \to \infty$. After considering a natural two-sided variant of the robot rendezvous problem already studied in Feintuch and Francis (2012a) and in particular proving a new quantified result in this case, we conclude by relating the robot rendezvous problem to a more realistic model of vehicle platoons.

1. Introduction

Consider a situation in which there are countably many robots (or perhaps ants, beetles, vehicles, etc.), indexed by the integers $\mathbb{Z}$, which at each time $t \geq 0$ occupy the respective positions $x_k(t)$, $k \in \mathbb{Z}$, in the complex plane. Suppose moreover that, for each $k \in \mathbb{Z}$ and each time $t \geq 0$, robot k moves in the direction of robot $k-1$ with speed equal to their separation, so that

$$\dot{x}_k(t) = x_{k-1}(t) - x_k(t), \quad k \in \mathbb{Z},\ t \geq 0. \tag{1.1}$$

We propose to investigate whether all of the robots necessarily converge to a mutual meeting, or *rendezvous*, point as $t \to \infty$, that is to say whether there exists $c \in \mathbb{C}$ such that $x_k(t) \to c$ as $t \to \infty$ uniformly in $k \in \mathbb{Z}$.

The problem is a natural extension of the corresponding question for finitely many robots, and in the finite case it is a simple matter to show that all robots converge exponentially fast to the centroid of their initial positions. However, since the actual rate of exponential convergence tends to zero as the size of the system grows this leaves open the question whether in the infinite case one should expect any rate of convergence, or even convergence for all initial constellations. Indeed, it was shown in Feintuch and Francis (2012a,b) that in the infinite setting there exist initial configurations of the robots which do not lead to convergence. The aim of this note is to revisit and extend a recent result due to the authors (Paunonen & Seifert, in press) giving a complete and simple characterisation of which initial configurations do and which do not lead to convergence. Loosely speaking, we show that the robots converge to the centroid of their initial positions whenever this is well-defined in a suitable sense, and do not converge otherwise. In addition, we present a detailed description of the rates of convergence of the robots. Thus our paper serves to further elucidate the similarities and differences between large finite systems and infinite systems. For further discussion of the relation between finite and infinite systems of the general kind considered here, see for instance (Curtain, Iftime, & Zwart, 2009).

Our approach is based on the asymptotic theory of C_0-semigroups and elements of ergodic theory, and the paper is organised as follows. Our first main result, giving a characterisation of those initial configurations leading to convergent solutions of the robot rendezvous problem, is presented in Section 2. In Section 3 we present a new proof of a quantified result from Paunonen and Seifert (in press), which provides an optimal estimate of the rate of convergence for initial configurations satisfying a certain condition, and in Section 4 we show how similar techniques lead to a

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new quantified result in a natural two-sided variant of the robot rendezvous problem considered in Feintuch and Francis (2012a). We conclude in Section 5 by describing a more realistic model which is representative of the general framework studied in depth in Paunonen and Seifert (in press).

2. Characterising 'good' initial constellations

We begin by introducing some preliminary notions. Let $\ell^\infty(\mathbb{Z})$ denote the space of doubly infinite sequences (x_k) satisfying $\sup_{k\in\mathbb{Z}}|x_k| < \infty$, endowed with the supremum norm

$$\|(x_k)\| = \sup_{k\in\mathbb{Z}}|x_k|, \quad (x_k) \in \ell^\infty(\mathbb{Z}).$$

Since we are interested in convergence of the solution $x(t) = (x_k(t))$, $t \geq 0$, with respect to the norm of $\ell^\infty(\mathbb{Z})$, it is natural to assume that the initial constellation $x_0 = (x_k(0))$ is an element of $\ell^\infty(\mathbb{Z})$, and we make this assumption throughout. We let S denote the right-shift operator on $\ell^\infty(\mathbb{Z})$, so that $S(x_k) = (x_{k-1})$ for all $(x_k) \in \ell^\infty(\mathbb{Z})$.

We say that an initial constellation x_0 in the robot rendezvous problem is *good* if there exist $c_k \in \mathbb{C}$, $k \in \mathbb{Z}$, such that the solution $x(t)$, $t \geq 0$, of (1.1) satisfies

$$\sup_{k\in\mathbb{Z}}|x_k(t) - c_k| \to 0, \quad t \to \infty.$$

In the finite case all initial constellations are good, and the robots all converge to the centroid of their initial positions. The following result shows that in the infinite robot rendezvous problem an initial constellation x_0 is good if and only if the translates $S^k x_0$, $k \geq 1$, under the right-shift operator S are Cesàro summable with respect to the norm of $\ell^\infty(\mathbb{Z})$, and that in this case the solution $x(t)$ of (1.1) converges to this Cesàro limit, which is necessarily a constant sequence, as $t \to \infty$. The result was originally obtained in Paunonen and Seifert (in press, Theorem 6.1) as a consequence of a more general result with a lengthy proof. Here we give a short and direct proof combining the main result of Feintuch and Francis with elementary facts from ergodic theory.

Theorem 1. *In the robot rendezvous problem (1.1), an initial constellation $x_0 = (x_k(0))$ is good if and only if there exists $c \in \mathbb{C}$ such that*

$$\sup_{k\in\mathbb{Z}}\left|\frac{1}{n}\sum_{j=1}^{n} x_{k-j}(0) - c\right| \to 0, \quad n \to \infty, \tag{2.1}$$

and if this is the case then

$$\sup_{k\in\mathbb{Z}}|x_k(t) - c| \to 0, \quad t \to \infty. \tag{2.2}$$

Proof. Let T denote the C_0-semigroup generated by $S - I$, so that $T(t) = \exp(t(S - I))$ for $t \geq 0$. Then the operators $T(t)$, $t \geq 0$, are uniformly bounded in operator norm and the solution of (1.1) is given by $x(t) = T(t)x_0$, $t \geq 0$. It follows from Feintuch and Francis (2012a, Theorem 3) that for initial constellations x_0 which lie in the range of $S - I$ we have $|x_k(t)| \to 0$ as $t \to \infty$ uniformly in $k \in \mathbb{Z}$. Since the semigroup T is uniformly bounded, the same conclusion holds for all initial constellations in the closure Y of this range. Next observe that the kernel Z of $S - I$ consists precisely of all constant sequences, and that such sequences are fixed by the semigroup. Let X denote the space of all initial constellations in $\ell^\infty(\mathbb{Z})$ which can be written (uniquely) as the sum of an element of Y and an element of Z. Then by the above observations all elements of X are good. By Arendt, Batty, Hieber, and Neubrander (2011, Proposition 4.3.1) the elements of X are also precisely those initial constellations x_0 for which the Cesàro means

$$\frac{1}{t}\int_0^t T(s)x_0\,ds, \quad t > 0,$$

converge in the norm of $\ell^\infty(\mathbb{Z})$ to a limit as $t \to \infty$. Since this is the case for any good initial constellation, X in fact coincides with the set of all good constellations. Moreover, it is clear that if $x_0 = y + z \in X$ with $y \in Y$ and $z \in Z$ being the constant sequence with entry $c \in \mathbb{C}$, then (2.2) holds. To finish the proof it suffices to observe that by Krengel (1985, Section 2.1, Theorem 1.3) the set X also coincides with the set of all initial constellations x_0 for which (2.1) is satisfied. □

It may be shown that condition (2.1) is satisfied for a wide range of initial constellations $x_0 = (x_k(0))$, for instance whenever $x_k(0) = c + y_k$, $k \in \mathbb{Z}$, where $|y_k| \to 0$ as $k \to \pm\infty$. In particular, the set of good initial constellation is stable under perturbations by sequences which converge to zero. Thus Theorem 1 strengthens (Feintuch & Francis, 2012a, Lemma 2). The result furthermore reveals the underlying reason for why the construction given in Feintuch and Francis (2012a, Section 3.5) leads to an initial constellation x_0 which is not good and in particular gives a simple way of constructing other examples, for instance by taking $x_0 = (x_k)$ to have entries $x_k = 0$ for $k \geq 0$ and, for $k < 0$, alternating blocks of zeros and ones having lengths which increase at suitable rates. Perhaps the most important contribution of Theorem 1 to the theory developed in Feintuch and Francis (2012a) is the observation that the correct topology in which Cesàro convergence of translates needs to be studied is not the topology of convergence in each entry but the norm topology of $\ell^\infty(\mathbb{Z})$.

We observe in passing that, even though it is argued in Feintuch and Francis (2012a,b) that the above setting for the robot rendezvous is the most realistic, the problem can also be studied with initial constellations lying in $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$; see Paunonen and Seifert (in press, Theorem 6.1). The upshot is that for $1 \leq p < \infty$ the only possible rendezvous point is the origin, and that all initial constellations are good if $1 < p < \infty$ but not when $p = 1$. The latter statement is an immediate consequence of the well-known fact that the right-shift operator S is mean ergodic on $\ell^p(\mathbb{Z})$ if and only if $1 < p < \infty$.

3. A quantified result

The following result is a quantified refinement of Theorem 1 and gives an estimate on the *rate* of convergence for initial constellations x_0 which satisfy a slightly stronger condition than (2.1). The result was originally obtained in Paunonen and Seifert (in press, Theorem 6.1). However, whereas the proof given in Paunonen and Seifert (in press) relies on direct estimates involving Stirling's formula, we present here a new and more elegant proof. In what follows, given two sequences $(a_n)_{n\geq1}$ and $(b_n)_{n\geq1}$ of non-negative numbers, we write $a_n = O(b_n)$ as $n \to \infty$ if there exists a constant $C > 0$ such that $a_n \leq Cb_n$ for all sufficiently large $n \geq 1$, and we use a similar notation for functions of a real variable.

Theorem 2. *In the robot rendezvous problem (1.1), if $x_0 = (x_k(0))$ is a good initial constellation such that*

$$\sup_{k\in\mathbb{Z}}\left|\frac{1}{n}\sum_{j=1}^{n} x_{k-j}(0) - c\right| = O\left(n^{-1}\right), \quad n \to \infty, \tag{3.1}$$

for some $c \in \mathbb{C}$, then

$$\sup_{k\in\mathbb{Z}}|x_k(t) - c| = O\left(t^{-1/2}\right), \quad t \to \infty.$$