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Brief paper

Nonlinear moving horizon estimation in the presence of bounded disturbances[☆]

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ABSTRACT

In this paper, we propose a new moving horizon estimator for nonlinear detectable systems. Similar to a recently proposed full information estimator, the corresponding cost function contains an additional *max*-term compared to more standard least-squares type approaches. We show that robust global asymptotic stability in case of bounded disturbances and convergence of the estimation error in case of vanishing disturbances can be established. Second, we show that the same results hold for a standard least-squares type moving horizon estimator, which so far has not been proven even in the full information estimation case. An additional advantage of the proposed estimators is that a suitable prior weighting appearing in the cost function can explicitly be determined offline, which is not the case in various existing approaches.

1. Introduction

Moving horizon estimation (MHE) is an optimization-based state estimation technique which has received an increasing amount of attention in recent years. At each time step, the current state estimate is determined by solving an optimization problem taking a number of past measurements into account. In order to obtain a good state estimate, ideally all past measurements from the initial time up to the current time should be considered, resulting in a so-called full information estimator. Since this is in general computationally intractable, in moving horizon estimation one instead uses only a fixed number N of past measurements. In recent years, both theoretical properties of various moving horizon estimation schemes as well as efficient computational methods for real-time implementation have been studied, see, e.g., Rawlings and Ji (2012), Rawlings and Mayne (2009) and Wynn, Vukov, and Diehl (2014).

In particular, it is of interest to establish (robust) stability properties of moving horizon estimation. To this end, in recent years various results have been obtained for different system classes, advancing from more restrictive or idealistic assumptions such as observability and no disturbances to less restrictive or more realistic cases such as detectability or the presence of bounded disturbances. For example, in Rao, Rawlings, and Mayne (2003), asymptotic stability of the estimation error is established for nonlinear observable systems without disturbances. For nonlinear detectable systems subject to asymptotically vanishing disturbances, robust global asymptotic stability (RGAS) and asymptotic convergence of the estimation error are established in Rawlings and Ji (2012) and Rawlings and Mayne (2009). In Alessandri, Baglietto, and Battistelli (2008) and Alessandri, Baglietto, Battistelli, and Zavala (2010), the authors propose a moving horizon estimation scheme for which a bounded estimation error can be guaranteed for the class of nonlinear observable systems subject to additive bounded disturbances. Under the assumption that a deterministic observer exists and is known, a moving horizon estimator is designed in Liu (2013) for nonlinear systems resulting in a bounded estimation error in case of bounded disturbances. Finally, for the general case of nonlinear detectable systems subject to bounded disturbances, a first step was recently taken in Ji, Rawlings, Hu, Wynn, and Diehl (2016), where a full information estimator was proposed which includes in the objective function an additional *max*-term compared to more standard least-squares type approaches. For this estimator, RGAS was established in Ji et al. (2016); this result was extended to more general conditions on the cost function in Hu, Xie, and You (2015). On the other hand, it could not be proven that the estimation

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error converges to zero if the disturbances asymptotically vanish[1] (compare Ji et al., 2016, Section IV.C).

The results presented in this paper improve on the existing literature in several ways. First, we propose for the first time a moving horizon estimator for which RGAS can be shown in the practically important case of general nonlinear detectable systems subject to bounded disturbances. To this end, we use an additional *max*-term in the objective function similar to Ji et al. (2016). Additionally, the proposed estimator enjoys the property that the estimation error converges to zero for asymptotically decaying disturbances. We note that in the general case of nonlinear detectable systems, one single estimator satisfying both these properties has not been known so far even in the full information estimator case as noted above. Second, we show that the same results can be obtained when using a standard least-squares type moving horizon estimator, i.e., without an additional *max*-term in the objective function. RGAS of such an estimator in case of bounded disturbances has been observed in practice in many publications and has thus been conjectured to hold (compare, e.g., Rawlings & Ji, 2012), but has not been proven so far even for a full information estimator. Furthermore, in order to establish robust stability properties of moving horizon estimators, it is crucial that the prior weighting in the cost function is chosen properly. In various of the existing MHE schemes such as Rao et al. (2003) and Rawlings and Mayne (2009), the necessary assumptions on the prior weighting are difficult to verify. In particular, it might not be possible to determine the prior weighting a priori, i.e., offline (compare Rawlings & Mayne, 2009, Assumption 4.17). An additional advantage of the MHE scheme proposed in this paper is that the assumption on the prior weighting can be verified a priori also in the general nonlinear case. Finally, we note that the detectability condition which we use is the same as the one used in the full information estimator case (Ji et al., 2016; Rawlings & Ji, 2012), which is slightly less restrictive compared to the one which was needed in the moving horizon case in Rawlings and Mayne (2009).

We close this section by noting that a preliminary version of parts of this paper has appeared in the conference proceedings (Müller, 2016). Besides a more comprehensive exposition of the subject, the main novelties of this paper compared to Müller (2016) are improved robust stability results in case that the system satisfies an exponential detectability condition, the establishment of RGAS in the presence of bounded disturbances for a standard least-squares type moving horizon estimator (without additional *max*-term), and a more exhaustive example section.

2. Preliminaries and setup

2.1. Notation

Let $\mathbb{I}_{[a,b]}$ denote the set of integers in the interval $[a, b] \subseteq \mathbb{R}$, and $\mathbb{I}_{\geq a}$ the set of integers greater than or equal to a. For $a \in \mathbb{R}$, $\lceil a \rceil$ is defined as the smallest integer greater than or equal to a. For $n \in \mathbb{I}_{\geq 1}$, I_n denotes the $n \times n$ identity matrix. For a vector $x = [x_1 \ \dots \ x_n]^T \in \mathbb{R}^n$ and $1 \leq p < \infty$, the p-norm is defined as $\|x\|_p := (\sum_{i=1}^p |x_i|^p)^{1/p}$, and $\|x\|_\infty := max_{i \in \mathbb{I}_{[1,n]}} |x_i|$. In the following, we abbreviate the Euclidean norm $\|x\|_2$ by $|x|$. Bold-face symbols denote sequences of finite or infinite length, i.e., $\boldsymbol{v} := \{v(t_1), \dots, v(t_2)\}$ for some $t_1, t_2 \in \mathbb{I}_{\geq 0}$ or $\boldsymbol{v} := \{v(0), v(1), \dots\}$, respectively. Let $\|\boldsymbol{v}\| := \sup_{t \in \mathbb{I}_{\geq 0}} |v(t)|$ denote the supremum norm of the sequence $\boldsymbol{v}$, $\|\boldsymbol{v}\|_{[a,b]} := \sup_{t \in \mathbb{I}_{[a,b]}} |v(t)|$, and $\|\boldsymbol{v}\|_{\geq a} := \sup_{t \in \mathbb{I}_{\geq a}} |v(t)|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$. A function $\alpha : \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{L}$ if α is nonincreasing and $\lim_{t \to \infty} \alpha(t) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \in \mathbb{I}_{\geq 0}$, and $\beta(r, \cdot)$ is of class $\mathcal{L}$ for each fixed $r \geq 0$.

2.2. Problem statement

We consider the state estimation problem for nonlinear discrete time systems of the form

$$
\begin{aligned}
x(t+1) &= f(x(t), w(t)), \qquad x(0) = x_0 \\
y(t) &= h(x(t)) + v(t)
\end{aligned}
\tag{1}
$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^m$, process disturbance $w \in \mathbb{R}^p$, and measurement noise $v \in \mathbb{R}^m$. In the following, we assume that the functions $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are continuous. The solution to system (1) at time t for initial condition x_0 and disturbance sequence $\boldsymbol{w} = \{w(0), w(1), \dots\}$ is denoted by $x(t; x_0, \boldsymbol{w})$, or simply by $x(t)$ if there is no ambiguity about x_0 and $\boldsymbol{w}$. Furthermore, we define $h_{\boldsymbol{w}}(\boldsymbol{x}) := \{h(x(0; x_0, \boldsymbol{w})), h(x(1; x_0, \boldsymbol{w})), \dots\}$. The process disturbance w and measurement noise v are unknown but assumed to be bounded. This means that we consider disturbance sequences $\boldsymbol{w} \in \mathbb{W}(w_{\max}) := \{\boldsymbol{w} : \|\boldsymbol{w}\| \leq w_{\max}\}$ and $\boldsymbol{v} \in \mathbb{V}(v_{\max}) := \{\boldsymbol{v} : \|\boldsymbol{v}\| \leq v_{\max}\}$ for some $w_{\max}, v_{\max} \geq 0$. Furthermore, the initial condition x_0 of system (1) is unknown. However, we assume that some prior knowledge $\bar{x}_0$ about the initial condition is available, and that the error of this prior estimate is bounded, i.e., $\bar{x}_0 \in \mathbb{X}_0(e_{\max}) := \{\bar{x}_0 : |x_0 - \bar{x}_0| \leq e_{\max}\}$ for some $e_{\max} \geq 0$.

The objective is to find, at each time t, an estimate $\hat{x}(t)$ of the current state $x(t)$, which will be done via moving horizon estimation (MHE) with some finite estimation horizon $N \in \mathbb{I}_{\geq 1}$. To this end, at each time $t \in \mathbb{I}_{\geq N}$, given the past N measurements $y(t-N), \dots, y(t-1)$, the following optimization problem is solved for some constants $\delta_1, \delta_2 > 0$ and $\delta \geq 0$:

$$
\begin{aligned}
&\underset{\chi(t-N|t), \boldsymbol{\omega}(t)}{\text{minimize}} \; J_N(\chi(t-N|t), \boldsymbol{\omega}(t)) \\
\text{s.t.} \quad & \chi(i+1|t) = f(\chi(i|t), \omega(i|t)), \\
& y(i) = h(\chi(i|t)) + \nu(i|t), \quad i \in \mathbb{I}_{[t-N, t-1]}
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
J_N(\chi(t-N|t), \boldsymbol{\omega}(t)) &:= \delta_1 \Gamma_{t-N}(\chi(t-N|t)) \\
&+ \delta_2 \sum_{i=t-N}^{t-1} \ell(\omega(i|t), \nu(i|t)) + \delta \, max_{i \in \mathbb{I}_{[t-N,t-1]}} \ell(\omega(i|t), \nu(i|t)).
\end{aligned}
\tag{3}
$$

In (2)–(3), $\boldsymbol{\omega}(t) := \{\omega(t-N|t), \dots, \omega(t-1|t)\}$ are the estimated process disturbances for time $t-N$ up to $t-1$, estimated at time t. Similarly, $\chi(i|t)$ and $\nu(i|t)$ denote estimated state and measurement noise variables for time i, estimated at time t. The stage cost ℓ penalizes the estimated process disturbances $\omega(i|t)$ and the fitting errors $\nu(i|t) = y(i) - h(\chi(i|t))$; conditions for a suitable choice of ℓ as well as for the prior weighting Γ_{t-N} are discussed below. For $t \in \mathbb{I}_{[0,N-1]}$ (i.e., until the estimation horizon is full), in (2) the optimization variables are replaced by $\chi(0|t)$ and $\boldsymbol{\omega}(t) := \{\omega(0|t), \dots, \omega(t-1|t)\}$, and the objective function J_N in (3) is replaced by[2]

$$
J_N(\chi(0|t), \boldsymbol{\omega}(t)) := \delta_1 \Gamma_0(\chi(0|t)) + \delta_2 \sum_{i=0}^{t-1} \ell(\omega(i|t), \nu(i|t)) + \delta \, max_{i \in \mathbb{I}_{[0,t-1]}} \ell(\omega(i|t), \nu(i|t)).
\tag{4}
$$

[1] In Hu et al. (2015), the authors establish convergence of the estimation error to zero, but only for the case where it is known *a priori* that the disturbances asymptotically vanish and this knowledge is used in the estimator design.

[2] If, as later in Section 3, the constant δ_1 in (3) depends on N, i.e., $\delta_1 = \delta_1(N)$, then $\delta_1(t)$ instead of δ_1 is used in (4). The same holds with respect to δ_2.