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A new discrete-time stabilizability condition for Linear Parameter-Varying systems☆

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A B S T R A C T

We introduce a new condition for the stabilizability of discrete-time Linear Parameter-Varying (LPV) systems in the form of Linear Matrix Inequalities (LMIs). A distinctive feature of the proposed condition is the ability to handle variation in both the dynamics as well as in the input matrix without resorting to dynamic augmentation or iterative procedures. We show that this new condition contains the existing poly-quadratic stabilizability result as a particular case. A numerical example illustrates the results.

1. Introduction and motivation

Consider the class of time-varying discrete-time linear systems of the form

$$x(k+1) = A(\xi(k))x(k) + B(\xi(k))u(k), \tag{1}$$

where $x \in \mathbb{R}^n$ and the matrices $A(\xi(k))$ and $B(\xi(k))$ are assumed to depend affinely on the time-varying parameter $\xi(k)$, which takes values in the unit simplex

$$\Xi = \left\{ \xi \in \mathbb{R}_+^N : \sum_{i=1}^{N} \xi_i = 1 \right\}.$$

The affine assumption means that matrices $A(\xi(k))$ and $B(\xi(k))$ can be written as

$$A(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) A_i, \qquad B(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) B_i.$$

In this paper, we are concerned with *stabilizability* by a *gain-scheduled* controller of the form:

$$u(k) = K(\xi(k))\, x(k) \tag{2}$$

using Linear Matrix Inequalities (LMIs). There are various examples of practical applications of the above model, including spacecraft control (Calloni, Corti, Zanchettin, & Lovera, 2012; Corti, Dardanelli, & Lovera, 2012), active suspension systems (Do, da Silva, Sename, & Dugard, 2011; Do, Sename, & Dugard, 2010), and the many other applications in Mohammadpour and Scherer (2012) and related references.

To the best of our knowledge, the most general necessary and sufficient stabilizability conditions for this class of time-varying systems that can still be expressed as LMIs are the ones from Daafouz and Bernussou (2001), which we reproduce in the next lemma.

Lemma 1 (*Daafouz & Bernussou, 2001*)**.** *Consider the time-varying discrete-time linear system of the form (1). Assume that $B_i = B$ for all $i = 1, \ldots, N$. The following statements are equivalent:*

(a) *System (1) is* poly − quadratically stabilizable*;*
(b) *There exist matrices X_i, L_i and $Q_i \succ 0$, $i = 1, \ldots, N$, such that*

$$\begin{bmatrix} X_i + X_i^T - Q_i & X_i^T A_i^T + L_i^T B^T \\ A_i X_i + B L_i & Q_j \end{bmatrix} \succ 0, \tag{3}$$

for all $i, j = 1, \ldots, N$.

Furthermore, if inequalities (3) are feasible the gain-scheduled state-feedback controller (2) with gains

$$K(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) K_i, \quad K_i = L_i X_i^{-1}, \tag{4}$$

poly-quadratically stabilizes the system (1).

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The above lemma makes use of the notion of *poly-quadratic stability*, in which stability of the time-varying system (1) is proved by constructing an affine *parameter-dependent* Lyapunov function (Gahinet, Apkarian, & Chilali, 1996) of the form

$$V(x(k), \xi(k)) = x(k)^T P(\xi(k)) x(k), \quad P(\xi(k)) = \sum_{i=1}^{N} \xi_i(k) P_i \succ 0. \tag{5}$$

In Lemma 1, if inequalities (3) are feasible, then $P_i = Q_i^{-1}$ provides such a Lyapunov function.

The main deficiency of the condition in Lemma 1 is the fact that the system cannot have time-variation in the input matrix B, hence the assumption $B(\xi(k)) = B$. Note that one cannot simply let the B's in inequalities (3) vary with i. In fact, a key element proved in Daafouz and Bernussou (2001) is the fact that inequalities (3) and $P_i = Q_i^{-1}$ imply that

$$\begin{bmatrix} P_i & (A_i + BK_i)^T P_j \\ P_j(A_i + BK_i) & P_j \end{bmatrix} \succ 0 \tag{6}$$

for all $i, j = 1, \ldots, N$ from which

$$\begin{bmatrix} P(\xi(k)) & (A(\xi(k)) + BK(\xi(k)))^T P(\xi(k+1)) \\ \star & P(\xi(k+1)) \end{bmatrix} \succ 0 \tag{7}$$

for all $\xi(k), \xi(k+1) \in \Xi$ after a convex combination, where the $\star$ notation stand for symmetric blocks omitted for brevity. In fact, inequality (7) can be taken as a definition of poly-quadratic stabilizability. If B and K vary with i then (6) no longer implies (7).

The main contribution of the present paper is to introduce the following design condition for the poly-quadratic stabilizability of system (1) which does not require the assumption that $B(\xi(k)) = B$ and recovers Lemma 1 as a particular case.

Theorem 2. *Consider the time-varying discrete-time linear system of the form (1). If there exist X_i, L_i, Y_i, Z_i and $Q_i \succ 0$, $i = 1, \ldots, N$ such that*

$$\begin{bmatrix} X_i + X_i^T - Q_i & X_i^T A_i^T & -L_i^T \\ A_i X_i & Q_j - R_{i,j} & B_i Z_j - Y_j^T \\ -L_i & Z_j^T B_i^T - Y_j & Z_j + Z_j^T \end{bmatrix} \succ 0, \tag{8}$$

where

$$R_{i,j} = B_i Y_j + Y_j^T B_i^T, \tag{9}$$

for all $i, j = 1, \ldots, N$, then the gain-scheduled state-feedback controller (2) with gain $K(\xi(k))$ as in (4) poly-quadratic stabilizes the system (1). Furthermore, if $B_i = B$ for all $i = 1, \ldots, N$, then the converse also holds.

As we will show in detail later in Section 4, it is guaranteed to hold whenever the one in Lemma 1 holds.

2. Comparison with existing results

It is important to point out that there have been other attempts to overcome the limitations of Daafouz and Bernussou (2001), e.g. Borges, Oliveira, Abdallah, and Peres (2008, 2010), Mao (2003) and Montagner, Oliveira, Leite, and Peres (2005). Refs. Borges et al. (2008, 2010) are BMIs (Bilinear Matrix Inequalities), hence not computationally attractive. The conditions presented in Mao (2003) and Montagner et al. (2005) are LMI based. Ref. Mao (2003) is concerned with poly-quadratically stabilizing robust controllers only. Ref. Montagner et al. (2005) deal with gain-scheduling but does not seem to recover the stabilizability results (Daafouz & Bernussou, 2001) in the case $B_i = B_j = B$ as possible with Theorem 2. In the case $B_i \neq B_j$ both Theorem 2 and Montagner et al. (2005) are sufficient and a direct comparison is not straightforward. However, Montagner et al. (2005) require checking $O(N^3)$ inequalities whereas Theorem 2 involves $O(N^2)$ inequalities. Furthermore, the gain scheduled controller (4) is linear in the time-varying parameter $\xi(k)$ whereas the one from Montagner et al. (2005) is rational.

Before proceeding, let us briefly discuss another alternative for incorporating time-variation in the input matrix in the discrete-time case. A standard way to handle time-variation in the input matrix B is to work with an augmented system, for example:

$$\tilde{x}(k+1) = \tilde{A}(\xi(k))\,\tilde{x}(k) + \tilde{B}\,\tilde{u}(k), \quad \tilde{x} = \begin{pmatrix} x \\ u \end{pmatrix}, \tag{10}$$

where $A(\xi)$ and B have as vertices the matrices

$$\tilde{A}_i = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B}_i = \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{11}$$

This approach remains popular, as attested by recent applications in spacecraft control (Calloni et al., 2012; Corti et al., 2012) and the design of active suspension systems (Do et al., 2011, 2010). However, we have shown in Pandey, Sehr, and de Oliveira (2016) that for quadratic continuous-time stabilizability and in Sehr, Pandey, and de Oliveira (submitted for publication) for quadratic continuous-time performance, that such augmentation is counter-productive, in the sense that quadratic stabilizability or performance of the augmented system by a gain-scheduled controller of the form (2) in fact implies existence of a *robust controller*, that is one in which $K(\xi(k)) = K$ is independent of the time-varying parameter, $\xi(k)$, with same stability and performance guarantees. A similar property also holds for discrete-time quadratic stabilizability but not for poly-quadratic stabilizability (Pandey et al., 2016). This is surprising since in discrete-time augmentation necessarily comes with an additional cost. Indeed, a controller of the form

$$\tilde{u}(k) = \tilde{K}(\xi(k))\,\tilde{x}(k) \tag{12}$$

that stabilizes the augmented system (10) corresponds to the dynamic and strictly proper controller with realization:

$$z(k+1) = K_u(\xi(k))\,z(k) + K_x(\xi(k))\,x(k), \quad u(k) = z(k), \tag{13}$$

where K_u and K_x are obtained from the augmented gain

$$\tilde{K}(\xi(k)) = \begin{bmatrix} K_x(\xi(k)) & K_u(\xi(k)) \end{bmatrix}. \tag{14}$$

Because it is strictly proper, it necessarily introduces an additional delay in the feedback loop. For this reason, we expect that a procedure that can directly handle variation in the input matrix B will lead to even better closed-loop performance as compared with controllers obtained through augmentation. Indeed, this is the case with the condition we propose in Theorem 2 as illustrated by the following comparative numerical example.

3. Comparative numerical example

Consider the following time-varying linear discrete-time system from de Oliveira, Bernussou, and Geromel (1999) with:

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0.8\alpha & -0.5\alpha & 0.2 & 0.03 + \alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B(\beta) = \begin{bmatrix} \beta \\ 0 \\ 1 - \beta \\ 0 \end{bmatrix}.$$

Our goal is to determine the largest $\gamma > 0$ such that the above discrete-time system can be stabilized for all $0 \le \beta \le 1$ and $|\alpha| \le \gamma$. This system can be put in the form (1) with 4 vertices. The maximum possible values of γ using different conditions in the literature are summarized in Table 1. We have compared: