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Stability and performance verification of optimization-based controllers ☆

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ABSTRACT

This paper presents a method to verify closed-loop properties of optimization-based controllers for deterministic and stochastic constrained polynomial discrete-time dynamical systems. The closed-loop properties amenable to the proposed technique include global and local stability, performance with respect to a given cost function (both in a deterministic and stochastic setting) and the $\mathcal{L}_2$ gain. The method applies to a wide range of practical control problems: For instance, a dynamical controller (e.g., a PID) plus input saturation, model predictive control with state estimation, inexact model and soft constraints, or a general optimization-based controller where the underlying problem is solved with a fixed number of iterations of a first-order method are all amenable to the proposed approach.

The approach is based on the observation that the control input generated by an optimization-based controller satisfies the associated Karush–Kuhn–Tucker (KKT) conditions which, provided all data is polynomial, are a system of polynomial equalities and inequalities. The closed-loop properties can then be analyzed using sum-of-squares (SOS) programming.

1. Introduction

This paper presents a computational approach to analyze closed-loop properties of optimization-based controllers for constrained polynomial discrete-time dynamical systems. Throughout the paper we assume that we are given an optimization-based controller that at each time instance generates a control input by solving an optimization problem parametrized by a function of the past measurements of the controlled system's output, and we ask about closed-loop properties of this interconnection. This setting encompasses a wide range of control problems including the control of a polynomial dynamical system by a linear controller (e.g., a PID) with an input saturation, output feedback model predictive control with inexact model and soft constraints, or a general optimization-based controller where the underlying problem is solved approximately with a fixed number of iterations of a first-order[1] optimization method. Importantly, the method verifies all KKT points; hence it can be used to verify closed-loop properties of optimization-based controllers where the underlying, possibly nonconvex, optimization problem is solved with a local method with guaranteed convergence to a KKT point only.

The closed-loop properties possible to analyze by the approach include: global stability and stability on a given subset, performance with respect to a discounted infinite-horizon cost (where we provide polynomial upper and lower bounds on the cost attained by the controller over a given set of initial conditions, both in a deterministic and a stochastic setting), the $\mathcal{L}_2$ gain from a given disturbance input to a given performance output (where we provide a numerical upper bound).

The main idea behind the presented approach is the observation that the KKT system associated to an optimization problem with polynomial data is a system of polynomial equalities and inequalities. Consequently, provided that suitable constraint qualification conditions hold (see, e.g., Peterson, 1973), the

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[1] By a first order optimization method we mean a method using only function value and gradient information, e.g., the projected gradient method (see Section 4.4).

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solution of this optimization problem satisfies a system of polynomial equalities and inequalities. Hence, the closed-loop evolution of a polynomial dynamical system controlled by an optimization-based controller solving at each time step an optimization problem with polynomial data can be seen as a difference inclusion where the successor state lies in a set defined by polynomial equalities and inequalities. This difference inclusion is then analyzed using sum-of-squares (SOS) techniques (see, e.g., Lasserre, 2009, Parrilo, 2003 for introduction to SOS programming).

The approach is based on the observation of Primbs (2001) who noticed that the KKT system of a constrained linear–quadratic optimization problem is a set of polynomial equalities and inequalities and used the S-procedure (see, e.g., Ghaoui, Feron, & Balakrishnan, 1994) to derive sufficient linear matrix inequality (LMI) conditions for a given linear MPC controller to be stabilizing. In this work we significantly extend the approach in terms of both the range of closed-loop properties analyzed and the range of practical problems amenable to the method. Indeed, our approach is applicable to general polynomial dynamical systems, both deterministic and stochastic, and allows the analysis not only of stability but also of various performance measures. The approach is not only applicable to an MPC controller with linear dynamics and a quadratic cost function as in Primbs (2001) but also to a general optimization-based controller, where the optimization problem may not be solved exactly, encompassing all the above-mentioned control problems.

This work is a continuation of Korda and Jones (2013) where the approach was used to analyze the stability of optimization-based controllers where the optimization problem is solved approximately by a fixed number of iterations of a first order method. The results of Korda and Jones (2013) are summarized in Section 4.4 of this paper as one of the examples that fit in the presented framework.

The paper is organized as follows. Section 2 gives a brief introduction to SOS programming. Section 3 states the problem to be solved. Section 4 presents a number of examples amenable to the proposed method. Section 5 presents the main verification results: Section 5.1 on global stability, Section 5.2 on stability on a given subset, Section 5.3 on performance analysis in a deterministic setting, Section 5.4 on performance analysis in a stochastic setting and Section 5.5 on the analysis of the $\mathcal{L}_2$ gain in a robust setting. Computational aspects are discussed in Section 6. Numerical examples are in Section 7 and some proofs are collected in the Appendix.

2. Sum-of-squares programming

Throughout the paper we will rely on sum-of-squares (SOS) programming, which allows us to optimize, in a convex way, over polynomials with nonnegativity constraints imposed over a set defined by polynomial equalities and inequalities (see, e.g., Lasserre, 2009, Parrilo, 2003 for more details on SOS programming). In particular we will often encounter optimization problems with constraints of the form

$$\mathcal{L}V(x) \geq 0 \quad \forall x \in \mathbf{K}, \tag{1}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, $\mathcal{L}$ a linear operator mapping polynomials to polynomials (e.g., a simple addition or a composition with a fixed function) and

$$\mathbf{K} = \{x \in \mathbb{R}^n \mid g(x) \geq 0,\ h(x) = 0\},$$

where the functions $g : \mathbb{R}^n \to \mathbb{R}^{n_g}$ and $h : \mathbb{R}^n \to \mathbb{R}^{n_h}$ are vector polynomials (i.e., each component is a polynomial). A *sufficient* condition for (1) to be satisfied is

$$\mathcal{L}V = \sigma_0 + \sum_{i=1}^{n_g} \sigma_i g_i + \sum_{i=1}^{n_h} p_i h_i, \tag{2}$$

where σ_0 and σ_i, $i = 1, \ldots, n_g$, are SOS polynomials and p_i, $i = 1, \ldots, n_h$, are arbitrary polynomials. A polynomial σ is SOS if it can be written as

$$\sigma(x) = \beta(x)^\top \mathcal{Q} \beta(x), \quad \mathcal{Q} \succeq 0, \tag{3}$$

where $\beta(x)$ is a vector of polynomials and $\mathcal{Q} \succeq 0$ signifies that $\mathcal{Q}$ is a positive semidefinite matrix. The condition (3) trivially implies that $\sigma(x) \geq 0$ for all $x \in \mathbb{R}^n$. Importantly, the condition (3) translates to a set of linear constraints and a positive semidefiniteness constraint and therefore is equivalent to a semidefinite programming (SDP) feasibility problem. In addition, the constraint (2) is affine in the coefficients of V, σ and p; therefore (2) also translates to an SDP feasibility problem and, crucially, it is possible to *optimize* over the coefficients of V (as long as they are affinely parametrized in the decision variables) subject to the constraint (2) using semidefinite programming.

In the rest of the paper when we say that a constraint of the form (1) is replaced by sufficient SOS constraints, then we mean that (1) is replaced with (2).

In addition we will often encounter optimization problems with objective functions of the form

$$\min/\max \int_{\mathbf{X}} V(x)\, dx, \tag{4}$$

where V is a polynomial and $\mathbf{X}$ a simple set (e.g., a box). The objective function is linear in the coefficients of the polynomials V. Indeed, expressing $V(x) = \sum_{i=1}^{n_\beta} v_i \beta_i(x)$, where $(\beta_i)_{i=1}^{n_\beta}$ are fixed polynomial basis functions and $(v_i)_{i=1}^{n_\beta}$ the corresponding coefficients, we have

$$\int_{\mathbf{X}} V(x)\, dx = \sum_{i=1}^{n_\beta} v_i \int_{\mathbf{X}} \beta_i(x)\, dx = \sum_{i=1}^{n_\beta} v_i m_i,$$

where the moments $m_i := \int_{\mathbf{X}} \beta_i(x)\, dx$ can be precomputed (in a closed form for simple sets $\mathbf{X}$). We see that the objective (4) is linear in the coefficients $(v_i)_{i=1}^{n_\beta}$ and hence optimization problems with objective (4) subject to the constraint (1) enforced via the sufficient constraint (2) translate to an SDP.

3. Problem statement

We consider the nonlinear discrete-time dynamical system

$$x^+ = f_x(x, u), \tag{5a}$$

$$y = f_y(x), \tag{5b}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ the control input, $y \in \mathbb{R}^{n_y}$ the output, $x^+ \in \mathbb{R}^{n_x}$ the successor state, $f_x : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ a transition mapping and $f_y : \mathbb{R}^{n_x} \to \mathbb{R}^{n_y}$ an output mapping. We assume that each component of f_x and f_y is a multivariate polynomial in (x, u) and x, respectively.

We assume that the system is controlled by a given set-valued controller

$$u \in \kappa(\mathbf{K}_s), \tag{6}$$

where $\kappa : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_u}$ is polynomial and

$$\mathbf{K}_s := \{\theta \in \mathbb{R}^{n_\theta} \mid \exists \lambda \in \mathbb{R}^{n_\lambda} \text{ s.t. } g(s, \theta, \lambda) \geq 0,\ h(s, \theta, \lambda) = 0\}, \tag{7}$$

where each component of the vector-valued functions $g : \mathbb{R}^{n_s} \times \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_\lambda} \to \mathbb{R}^{n_g}$ and $h : \mathbb{R}^{n_s} \times \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_\lambda} \to \mathbb{R}^{n_h}$ is a polynomial in the variables (s, θ, λ). The set $\mathbf{K}_s$ is parametrized by the output of a dynamical system

$$z^+ = f_z(z, y), \tag{8a}$$

$$s = f_s(z, y), \tag{8b}$$

where $f_z : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_z}$ and $f_s : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_s}$ are polynomials. The problem setup is depicted in Fig. 1. In the rest of the paper we develop a method to analyze the closed-loop stability and performance of this interconnection. Before doing that we present several examples which fall into the presented framework.