Automatica 78 (2017) 14–19

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Partial stability for nonlinear model predictive control ☆

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ARTICLE INFO

Article history:
Received 8 June 2015
Received in revised form
30 August 2016
Accepted 11 November 2016

Keywords:
Nonlinear model predictive control
Partial stability
Parameter estimation

ABSTRACT

While the theory of stability for nonlinear model predictive control (NMPC) has been under extensive study, partial stability is apparently overlooked. In this paper we apply the concept of partial stability in order to extend several results for the stability analysis of NMPC and investigate the behavior of NMPC for dynamic systems with uncertain parameters. Partial stability for NMPC is established without using terminal costs and terminal constraints. Numerical examples are provided to illustrate the obtained results.

1. Introduction

Nonlinear Model Predictive Control (NMPC) has become the predominant advanced control methodology in recent years. Together with numerical methods, stability analysis for NMPC is widely studied. Stability criteria have been devised for various NMPC scenarios, including infinite horizon control, receding horizon control with zero terminal constraints, Mayne and Michalska (1990) and Michalska and Mayne (1993); quadratic terminal costs with regional terminal constraints, Chen and Allgöwer (1998); general terminal costs which are control Lyapunov functions, Fontes (2001) and Jadbabaie and Hauser (2005). See also Findeisen, Imsland, Allgöwer, and Foss (2003), Grüne and Pannek (2011) and Rawlings and Mayne (2009) for comprehensive overviews. Very recent studies concern stability of NMPC without constraints, e.g., Grüne (2013), Jadbabaie, Primbs, and Hauser (2001) and Reble and Allgöwer (2012). We consider the following NMPC setup. The dynamic system is given by:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \geq 0, \tag{1}$$

where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^{n_u}$, with $0 \in U$, $0 \in X$ and $f(t, 0, 0) = 0$ for all t. Choose $T > 0$ as a prediction horizon. For each t, x^0, consider the following optimal control problem (OCP)

$$\begin{aligned} \min_{x(\cdot),u(\cdot)} \quad & \int_t^{t+T} L(x(\tau), u(\tau))d\tau, \\ \text{s.t.} \quad & \dot{x}(\tau) = f(\tau, x(\tau), u(\tau)), \quad \tau \geq t, \qquad (P_T(t, x^0)) \\ & x(t) = x^0, \\ & u(\tau) \in U, \quad x(\tau) \in X, \quad \forall \tau \geq t. \end{aligned}$$

Control and state bounds can be enforced via the sets U and X, which we assume to be closed. The function $L : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^+$ is often assumed to satisfy

$$L(0, 0) = 0, \qquad L(x, u) \geq \gamma(\|x\|) \quad \forall x \in X, u \in U, \tag{2}$$

with some $\gamma \in \mathcal{K}_\infty$ to be defined in Section 2.

For each t, x^0, suppose that $(P_T(t, x^0))$ has a unique solution with the optimal controls and states $u^*_{T,t,x^0}(\tau)$, $x^*_{T,t,x^0}(\tau)$ and the optimal value $V_T(t, x^0)$. Take a sampling time $T_s < T$ and the grid $t_k = kT_s$, $k = 0, 1, 2, \ldots$. We consider the following NMPC scheme with sampling.

0. Set $k = 0$.
1. Estimate state $\hat{x}(t_k)$.
2. Solve $(P_T(t_k, \hat{x}(t_k)))$.
3. Apply $u(t) = u^*_{T,t_k,\hat{x}(t_k)}(t)$, $t \in [t_k, t_{k+1})$ to process.
4. Set $k = k + 1$ and go to 1.

For *nominal stability* analysis of NMPC schemes, it is assumed that the estimates $\hat{x}(t_k)$ coincide with the true values. Our setup does not impose any terminal costs or terminal constraints needed in Chen and Allgöwer (1998), Findeisen et al. (2003), Mayne and Michalska (1990) and Michalska and Mayne (1993). In fact, it is the state-of-the-art setup which is recently studied in Grimm, Messina,

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Zongli Lin under the direction of Editor Andrew R. Teel.

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http://dx.doi.org/10.1016/j.automatica.2016.11.047

Tuna, and Teel (2005), Grüne (2013) and Reble and Allgöwer (2012). However, assumption (2) which relies on the coerciveness of L with respect to the complete x may be too restrictive, e.g., for economic NMPC, which becomes more and more popular, Diehl, Amrit, and Rawlings (2011) and Grüne (2013). While there is rich literature on nominal stability, the problem of *partial stability* for NMPC seems to be overlooked. In this paper, we will use partial stability to weaken assumption (2). Moreover, for systems with uncertain parameters and disturbances, the estimates of uncertain parameters may not converge to the true values. It is often the case that, when the states enter a certain region, they provide less and less information to estimate the parameters. One of our goals is to apply the theory of partial stability to the stability analysis for NMPC in such cases. Also when controlling systems under disturbances, we must ensure that the states are (asymptotically) stable provided that the disturbances lie within some suitable region. This is related to the problem of robust control and we will present a treatment of this problem in the framework of partial stability.

The problem of partial stability has been investigated in the theory of differential equations. It was initiated by Lyapunov in 1893 but extensively studied only later in the 1960s. Vorotnikov (1998) presents a comprehensive study about this subject. Besides the traditional framework of stability analysis for linear as well as nonlinear systems, Vorotnikov (1998) also considers problems of stabilization in connection with optimal control problems.

The paper is organized as follows. The concepts of partial stability for differential equations are presented in Section 2. We then establish partial stability for NMPC without terminal costs and terminal constraints in Section 3. In Section 4 we discuss the behavior of NMPC for dynamic systems with uncertain parameters and provide an illustrative numerical example. We complement the paper with an Appendix for the analytic investigation of the system considered in Section 4.

For convenience, we introduce some notation that we use throughout the paper: By $\boldsymbol{\partial} : \mathbb{R}^n \to \mathbb{R}^m$ we denote a continuous function, which we call the *partiality*-mapping of a vector $x \in \mathbb{R}^n$, e.g., the mapping of x onto its last m components $\boldsymbol{\partial}x = (x_{n-m+1}, x_{n-m+2}, \dots, x_n)^T$. In addition, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers.

2. Partial stability for differential equations

We consider the following initial value problem (IVP)

$$\begin{aligned} \dot{x}(t) &= \phi(t, x(t)), \\ x(t_0) &= x^0, \end{aligned} \tag{3}$$

where $\phi : \mathbb{R}^+ \times D \to \mathbb{R}^n$ is continuous and locally Lipschitz continuous with respect to x. Here D is an open connected subset of $\mathbb{R}^n$ that contains the origin $0 \in \mathbb{R}^n$. The solution of the IVP (3) is denoted by $x(t; x^0, t_0)$ or $x(t)$ if the initial conditions are already specified. Without loss of generality, we assume that $x^* = x(t; 0, 0) = 0$ for all $t \geq 0$, i.e., $\phi(t, 0) = 0$.

Definition 1 (*Partial Stability, Vorotnikov, 1998*)**.** The solution $x^* = 0$ of (3) is said to be $\boldsymbol{\partial}$-stable if for each $t_0 \geq 0$, $\varepsilon > 0$, there exists $\delta(t_0, \varepsilon) > 0$ such that for all x^0 with $\|x^0\| < \delta(t_0, \varepsilon)$, we have

$$\|\boldsymbol{\partial}x(t; x^0, t_0)\| < \varepsilon \quad \text{for all } t \geq t_0.$$

Definition 2 (*Vorotnikov, 1998*)**.** The solution $x^* = 0$ of (3) is said to be $\boldsymbol{\partial}$-asymptotically stable if it is $\boldsymbol{\partial}$-stable and for each t_0, there exists $\delta_0(t_0) > 0$ such that for any x^0 with $\|x^0\| < \delta_0(t_0)$, we have

$$\lim_{t\to\infty} \boldsymbol{\partial}x(t; x^0, t_0) = 0.$$

We now introduce some important classes of functions which make the investigation of the stability of IVP (3) convenient. Define for $R > 0$

$$\mathcal{K}_R = \big\{\alpha : [0, R) \to \mathbb{R}^+,\ \alpha(0) = 0,\\ \alpha \text{ is continuous, strictly increasing}\big\},$$

with the additional requirement of $\lim_{r\to\infty} \alpha(r) = \infty$ if $R = \infty$. For each $\alpha \in \mathcal{K}_R$, there exists the inverse of α, denoted by $\alpha^{-1} : [0, \bar{\alpha}) \to [0, R)$ with $\bar{\alpha} = \lim_{r\to R^-} \alpha(r) \in \mathbb{R}^+ \cup \{+\infty\}$.

3. Partial stability for NMPC

By applying partial stability, we can weaken assumption (2). In fact, we assume that L only satisfies

$$L(0, 0) = 0, \quad L(x, u) \geq \gamma(\|\boldsymbol{\partial}x\|)\ \forall x \in X, u \in U, \tag{4}$$

with some $\gamma \in \mathcal{K}_\infty$. For nonautonomous systems, to ensure that $V_T(t, x^0) \geq \alpha_T(\|\boldsymbol{\partial}x^0\|)$, we suppose:

(H) There exist $T > 0$ and an $\alpha_T \in \mathcal{K}_\infty$ such that for any t, x^0 and $u(\tau), x(\tau)$ satisfying (1) with $x(t) = x^0$, we have

$$\int_t^{t+T} L(x(\tau), u(\tau))d\tau \geq \alpha_T\left(\|\boldsymbol{\partial}x^0\|\right).$$

Lemma 3. *Suppose that $0 \leq T_1 \leq T_2$. Then for any $t \geq 0, x^0 \in X$ we have $V_{T_2}(t, x^0) \geq V_{T_1}(t, x^0)$.*

Proof. The control $u^*_{T_2,t,x^0}(\tau)$ for $\tau \in [t, t + T_1]$ is feasible for $(P_{T_1}(t, x^0))$. Hence

$$\begin{aligned} V_{T_2}(t, x^0) &= \int_t^{t+T_2} L(x^*_{T_2,t,x^0}(\tau), u^*_{T_2,t,x^0}(\tau))d\tau \\ &\geq \int_t^{t+T_1} L(x^*_{T_2,t,x^0}(\tau), u^*_{T_2,t,x^0}(\tau))d\tau \geq V_{T_1}(t, x^0). \end{aligned}$$

Let us denote by $x(\tau)$, $u(\tau)$ the solution produced by the NMPC scheme with the initial states $x(0)$. We define $\tilde{L}_k(\tau) = L(x^*_{T,t_k,x(t_k)}(\tau), u^*_{T,t_k,x(t_k)}(\tau))$ for $\tau \in [t_k, t_k + T]$ and $\tilde{L} : [0, \infty) \to \mathbb{R}^+$ piecewise by

$$\tilde{L}(\tau) = L(x^*_{T,t_k,x(t_k)}(\tau), u^*_{T,t_k,x(t_k)}(\tau)) \quad \text{for } \tau \in [t_k, t_{k+1}).$$

Here is the main result of the paper:

Theorem 4. *If* (H) *is satisfied, V_T is continuous and there exist $\beta \in (0, 1]$ and $k_0 \geq 0$ such that for all $k \geq k_0$, it holds that*

$$V_T(t_{k+1}, x(t_{k+1})) \leq V_T(t_k, x(t_k)) - \beta \int_{t_k}^{t_{k+1}} \tilde{L}(\tau)d\tau. \tag{5}$$

Then the closed-loop system produced by the NMPC scheme is $\boldsymbol{\partial}$-asymptotically stable.

Proof. In the following we consider indices $k \geq k_0$. Define the function

$$V(t) = V_T(t_k, x(t_k)) - \beta \int_{t_k}^{t} \tilde{L}(\tau)d\tau$$

piecewise for $t \in [t_k, t_{k+1})$. By the Bellman dynamic programming principle (DP),

$$V_{T-t+t_k}(t, x(t)) = V_T(t_k, x(t_k)) - \int_{t_k}^{t} \tilde{L}(\tau)d\tau.$$