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Brief paper

A multiple-comparison-systems method for distributed stability analysis of large-scale nonlinear systems ☆

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ABSTRACT

Lyapunov functions provide a tool to analyze the stability of nonlinear systems without extensively solving the dynamics. Recent advances in sum-of-squares methods have enabled the algorithmic computation of Lyapunov functions for polynomial systems. However, for general large-scale nonlinear networks it is yet very difficult, and often impossible, both computationally and analytically, to find Lyapunov functions. In such cases, a system decomposition coupled to a vector Lyapunov functions approach provides a feasible alternative by analyzing the stability of the nonlinear network through a reduced-order comparison system. However, finding such a comparison system is not trivial and often, for a nonlinear network, there does not exist a single comparison system. In this work, we propose a multiple comparison systems approach for the algorithmic stability analysis of nonlinear systems. Using sum-of-squares methods we design a scalable and distributed algorithm which enables the computation of comparison systems using only communications between the neighboring subsystems. We demonstrate the algorithm by applying it to an arbitrarily generated network of interacting Van der Pol oscillators.

1. Introduction

A key to maintaining the successful operation of real-world engineering systems is to analyze the stability of the systems under disturbances. Lyapunov functions methods provide powerful tools to directly certify stability under disturbances, without solving the complex nonlinear dynamical equations (Haddad & Chellaboina, 2008; Lyapunov, 1892). However for a general nonlinear system, there is no universal expression for Lyapunov functions. Recent advances in sum-of-squares (SOS) methods and semi-definite programming (SDP), Papachristodoulou et al. (2013), Prajna, Papachristodoulou, Seiler, and Parrilo (2005) and Sturm (1999), have enabled the algorithmic construction of polynomial Lyapunov functions for nonlinear systems that can be expressed as a set of polynomial differential algebraic equations (Chesi, 2011; Tan, 2005). Unfortunately, such computational methods suffer from scalability issues and, in general, become intractable as the system size grows (Anderson, Chang, & Papachristodoulou, 2011). For this reason more tractable alternatives to SOS optimization have been proposed. One such approach, known as DSOS and SDSOS optimization, is significantly more scalable since it relies on linear programming and second order cone programming (Ahmadi & Majumdar, 2014). A different approach chooses Lyapunov functions with a chordal graphical structure in order to convert the semidefinite constraints into an equivalent set of smaller semidefinite constraints which can be exploited to solve the SDP programs more efficiently (Mason & Papachristodoulou, 2014). Nevertheless, the increased scalability decreases performance since both approximations are usually more conservative than SOS approaches.

Despite these computational advances, global analysis of large-scale systems remains problematic when computational and communication costs are considered. Often, a decomposition–aggregation approach offers a scalable distributed computing framework, together with a flexible analysis of structural perturbations (Šiljak, 1991) and decentralized control designs (Šiljak, 1978), as required by the locality of perturbations. Thus, for large-scale systems, it is often useful to model the system as a network of

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small interacting subsystems and study the stability of the full interconnected system with the help of the Lyapunov functions of the isolated subsystems. For example, one approach is to construct a scalar Lyapunov function expressed as a weighted sum of the subsystem Lyapunov functions and use it to certify stability of the full system (Araki, 1978; Michel, 1983; Šiljak, 1972; Weissenberger, 1973). However, such a method requires centralized computations and does not scale well with the size of the network. Alternatively, methods based on vector Lyapunov functions, Bailey (1966) and Bellman (1962), are computationally very attractive due to their parallel structure and scalability, and have generated considerable interest in recent times (Karafyllis & Papageorgiou, 2015; Kundu & Anghel, 2015a,c; Xu, Wang, Hong, Jiang, & Xu, 2016). However, applicability of these methods to large-scale nonlinear systems with guaranteed rate of convergence still remains to be explored. For example, in Kundu and Anghel (2015a,c) the authors consider asymptotic stability while the works in Karafyllis and Papageorgiou (2015) and Xu et al. (2016) are demonstrated on small-scale systems.

Inspired by the results on comparison systems, Beckenbach and Bellman (1961), Brauer (1961) and Conti (1956), it has been observed that the problem of stability analysis of an interconnected nonlinear system can be reduced to the stability analysis of a linear dynamical system (or, 'single comparison system') whose state space consists of the subsystem Lyapunov functions. Success in finding such stable linear comparison system then guarantees exponential stability of the full interconnected nonlinear system. However, for a given interconnected system, computing these comparison systems still remained a challenge. In absence of suitable computational tools, analytical insights were used to build those comparison systems, such as trigonometric inequalities in power systems networks (Jocic, Ribbens-Pavella, & Šiljak, 1978). In a recent work (Kundu & Anghel, 2015b), SOS-based direct methods were used to compute the single comparison system for generic nonlinear polynomial systems, with some performance improvements over the traditional methods. However there are major challenges before such a method can be used in large-scale systems. For example, it is generally difficult to construct a single comparison system that can guarantee stability under a wide set of disturbances. Also, while Kundu and Anghel (2015b) present a decentralized analysis where the computational burden is shared between the subsystems, the scalability of the analysis is largely dependent on the cumulative size of the neighboring subsystems.

In this article we present a novel conceptual and computational framework which generalizes the single comparison system approach into a sequence of stable comparison systems, that collectively ascertain stability, while also offering better scalability by parallelizing the subsystem-level SOS problems. The set of multiple comparison systems are to be constructed adaptively in real-time, after a disturbance has occurred. With the help of SOS and semi-definite programming methods, we develop a fully distributed, parallel and scalable algorithm that enables computation of the comparison systems under a disturbance, with only minimal communication between the immediate neighbors. While this approach is applicable to any generic dynamical system, we choose an arbitrarily generated network of modified[1] Van der Pol oscillators (Van der Pol, 1926) for illustration. Under a disturbance, the subsystems communicate with their neighbors to algorithmically construct a set of multiple comparison systems, the successful construction of which can certify stability of the network. The rest of this article is organized as follows. Following some brief background in Section 2 we describe the problem in Section 3. We present the traditional approach to single comparison systems and an SOS-based direct method of computing the comparison systems in Section 4. In Section 5, we introduce the concept of multiple comparison systems, and propose a parallel and distributed algorithmic construction of the comparison systems in real-time. We demonstrate an application of this algorithm to a network of Van der Pol oscillators in Section 6, before concluding the article in Section 7.

[1] Parameters are chosen to make the equilibrium point stable.

2. Preliminaries

Let us consider the dynamical system

$$\dot{x}(t) = f(x(t)), \quad t \geq 0,\ x \in \mathbb{R}^n,\ f(0) = 0, \tag{1}$$

with an equilibrium at the origin,[2] and $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. Let us use $|\cdot|$ to denote both the Euclidean norm (for a vector) and the absolute value (for a scalar).

Definition 1. The equilibrium point at the origin is said to be asymptotically stable in a domain $\mathscr{D} \subseteq \mathbb{R}^n$, $0 \in \mathscr{D}$, if $\lim_{t\to\infty} |x(t)| = 0$ for every $|x(0)| \in \mathscr{D}$, and it is exponentially stable if there exists $b,\ c > 0$ such that $|x(t)| < ce^{-bt}\,|x(0)|\ \forall t \geq 0$, for every $|x(0)| \in \mathscr{D}$.

Theorem 1 (*Lyapunov, 1892, Khalil, 1996, Thm. 4.1*). *If there exists a domain $\mathscr{D} \subseteq \mathbb{R}^n$, $0 \in \mathscr{D}$, and a continuously differentiable positive definite function $\tilde{V} : \mathscr{D} \to \mathbb{R}_{\geq 0}$, i.e. the 'Lyapunov function' (LF), then the origin of (1) is asymptotically stable if $\nabla\tilde{V}^T f(x)$ is negative definite in $\mathscr{D}$, and is exponentially stable if $\nabla\tilde{V}^T f(x) \leq -\alpha\,\tilde{V}\ \forall x \in \mathscr{D}$, for some $\alpha > 0$.*

The region of attraction (ROA) of the stable equilibrium point at origin can be (conservatively) estimated as Genesio, Tartaglia, and Vicino (1985)

$$\mathscr{R} := \{x \in \mathscr{D}\,|\,V(x) \leq 1\}, \quad \text{with } V(x) = \tilde{V}(x)/\gamma^{\max}, \tag{2a}$$

$$\text{where } \gamma^{\max} := \max\left\{\gamma \;\middle|\; \left\{x \in \mathbb{R}^n \,\middle|\, \tilde{V}(x) \leq \gamma\right\} \subseteq \mathscr{D}\right\}, \tag{2b}$$

i.e. the boundary of the ROA is estimated by the unit level-set of a suitably scaled LF $V(x)$. Relatively recent studies have explored how sum-of-squares (SOS) based methods can be utilized to find LFs by restricting the search space to SOS polynomials (Anghel, Milano, & Papachristodoulou, 2013; Jarvis-Wloszek, 2003; Parrilo, 2000; Tan, 2006). Let us denote by $\mathbb{R}[x]$ the ring of all polynomials in $x \in \mathbb{R}^n$.

Definition 2. A multivariate polynomial $p \in \mathbb{R}[x],\ x \in \mathbb{R}^n$, is a sum-of-squares (SOS) if there exist some polynomial functions $h_i(x)$, $i = 1 \ldots s$ such that $p(x) = \sum_{i=1}^{s} h_i^2(x)$. We denote the ring of all SOS polynomials in $x \in \mathbb{R}^n$ by $\Sigma[x]$.

Checking if $p \in \mathbb{R}[x]$ is an SOS is a semi-definite problem which can be solved with a MATLAB® toolbox SOSTOOLS (Papachristodoulou et al., 2013; Prajna et al., 2005) along with a semidefinite programming solver such as SeDuMi (Sturm, 1999). The SOS technique can be used to search for polynomial LFs by translating the conditions in Theorem 1 to equivalent SOS conditions (Chesi, 2010; Jarvis-Wloszek, 2003; Papachristodoulou et al., 2013; Papachristodoulou & Prajna, 2005; Prajna et al., 2005; Wloszek, Feeley, Tan, Sun, & Packard, 2005). An important result from algebraic geometry, called Putinar's Positivstellensatz theorem (Lasserre, 2009; Putinar, 1993), helps in translating the SOS conditions into SOS feasibility problems. The Putinar's Positivstellensatz theorem states (see Lasserre, 2009, Ch. 2):

[2] Note that by shifting the state variables any equilibrium point of interest can be moved to the origin.